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Classification of models of Jonsson theories regarding to cosemantic equivalence

Annotation
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Relevance of the research topic. The main research question in this thesis is the classification of models of Jonsson theories regarding to cosemantic equivalence. In addition, the properties of Jonsson independence are investigated in the forking language, defined as a binary relation on the set of special types and some subsets of the semantic model of a fixed Jonsson theory. The properties of lattices of existential formulas of Δ-Jonsson theories and their relation to well-known issues in model theory are also considered. In particular, the relationship between the perfectness of such theories and the properties of the lattice of equivalence classes of existential formulas with respect to this theory is considered.

Jonsson theories form a rather wide subclass of inductive theories, and it follows from the definition of Jonsson theories that, generally speaking, they are not complete. These theories satisfy natural requirements, such as inductance, co-embedding property and amalgam. Jonsson theories, for example, are theories of such classical algebraic systems as fields of fixed characteristic, groups, Abelian groups, various types of rings, Boolean algebras, lattices, polygons. T.G. Mustafin found a description of generalized Jonsson theories, which shows the relationship between complete theories, Jonsson theories and generalized Jonsson theories. Yeshkeyev A.R. continued the research of Jonsson theories regarding the various model-theoretical properties of their companions, including *J*-stability, was continued. In particular, in the framework of the study of Jonsson theories, an important concept such as forking, which was previously defined by S. Shelah, was redefined and is one of the main means of modern technique of Model Theory for the classification of complete theories.

Recently, a number of works have appeared that deal with problems similar to those arising in the study of Jonsson theories, i.e. those who are not complete but with some additional properties (for example, amalgam property, joint embedding property, insularity with respect to special subclasses, sometimes not elementary). So, for example, A. Pillay in one of his works examines the properties of simple theories that arise when studying the class of existentially closed models of an arbitrary universal. In the works of I. Ben-Yaacov, a positive model theory was introduced and in its framework so-called CATs were considered, which generalize the corresponding objects studied in the work of A. Pillay. Yeshkeyev A.R. new classes of positive Jonsson theories defined and positive Jonsson analogues of some works of V. Weispfenning, M. Morley, P. Lindström, and C. Ryll-Nardzewski were obtained. It should be noted that there are various regular ways of moving from an arbitrary theory to a Jonsson theory, which preserves the original class of existentially closed models. One of these methods is the morleization of theory. Thus, the study of model-theoretical properties of Jonsson theories is an urgent task, both in model theory and in universal algebra, and the questions concerning the study of Jonsson theories exactly relate in essence to the problems of the so-called "eastern" model theory, which , as a rule, deals with incomplete theories.

The aim of this work is to study of the property of cosemantic for algebraic structures satisfying Jonsson's conditions. In particular, the work deals with the questions classification of the theory of Abelian groups and modules with respect to the concepts of cosemantic and the Schröder-Bernstein property. In addition, we study the properties of Jonsson forking and independence for the class of the Jonsson spectrum of an arbitrary model of signature.

The objects of research are Jonsson theories and their classes of models. In particular, in the work are considered the model-theoretical questions of the theory of abelian groups and R-modules, and their class of existentially closed models, which will be elementary due to the perfectness of theories of abelian groups and R-modules in the case when the ring is coherent.

Research methods. The main research methods in this dissertation include the general methods of the classical model theory associated with the study of complete theories, as well as methods of universal algebra. In addition, the semantic method for the study of Jonsson theories is regularly used, which consists in transfer the elementary properties of the first order of the center of the Jonsson theory to the Jonsson theory itself. In the case when direct transfer from the Jonsson theory to its center is possible, as a rule, we work, even in the imperfect case, only with the class of existentially closed models.

The scientific novelty of the research topic. This topic is completely new, all the results obtained and published have no analogues due to the generality formulation of the problems of this dissertation.

Research objectives. Due to the fact that Jonsson theories, generally speaking, are not complete, the apparatus for studying such theories is undeveloped and the first stage of the study is to redefine the main terminological base of the classical model theory associated with solved and unsolved problems defined in the frame of the study of complete theories. Further, using the above research methods, we try to obtain analogues of the results that are true for the center of the Jonsson theory, which is a complete theory in the framework of the study of the Jonsson theory itself. In the absence of analogues, we try to find a more general formulation of the problem, which covers the conditions that are true both for the center and for the Jonsson theory itself.

Theoretical and practical value of the work. The work is theoretical and applied. The theoretical results obtained can be used in further studies of model-theoretical properties of Jonsson theories and their classes of models in classical model theory and universal algebra. The applied value of the results obtained can be applied in all areas of mathematics, where it is possible to interpret an abelian group or module. For example, since linear space is a special case of a module, an important example is the linear space of differential operators, which describes the physical properties of the corresponding important natural processes.

Approbation of the received results. The main results of the dissertation were reported and discussed at the following seminars and conferences:

– the international scientific conference "Theoretical and applied problems of mathematics, mechanics and computer science" (Kazakhstan, Karaganda, June 12-14, 2014);

– the international scientific conference “Algebra, analysis, differential equations and their applications”, dedicated to the 60th anniversary of Academician of the National Academy of Sciences of the Republic of Kazakhstan Dzhumadildaev Askar Serkulovich (Kazakhstan, Almaty, April 8-9, 2016);

– European Summer Meeting of the Association for Symbolic Logic “Logic Colloquium 2016” (United Kingdom, Leeds, July 31-August 6, 2016);

– 6th World Congress and School on Universal Logic (France, Vichy, June 16-26, 2018);

– the international scientific conference "Theoretical and applied questions of mathematics, mechanics and computer science", dedicated to the 70th anniversary of the doctor of physical and mathematical sciences, professor Ramazanov Murat Ibraevich (Kazakhstan, Karaganda, June 12-13, 2019);

– the 16th Asian Logic Conference (Kazakhstan, Nur-Sultan, June 17-21, 2019);

– Scientific seminar of the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin, supervisor Doctor of Physical and Mathematical Sciences, Professor A.R. Yeshkeyev (Buketov Karaganda State University).

Publications. The main results of the dissertation were published in 14 papers, of which 2 articles were published in a journal included in the zbMath and Scopus databases, 4 articles were published in journals recommended by the Committee for Control in Education and Science of the Ministry of Education of the Republic of Kazakhstan, and 8 scientific work were published in materials of international scientific conferences.

Provisions to be defended. The following main results of the dissertation research are submitted for defense:

1) the necessary and sufficient conditions are found what the Jonsson theory of Abelian groups to admit the JSB property;

2) a criterion is obtained for the cosemanticity of Abelian groups, which is a Jonsson version of W. Szmielew’s theorem on the elementary classification of Abelian groups;

3) a criterion for the cosemanticity of modules is obtained, which is a refinement of Monk's theorem on elementary equivalence of modules in the framework of the study of Jonsson module theories;

4) defined the concept of forking for the existential types of a some semantic model of an arbitrary Jonsson theory based on the axioms of the definition of forking in the classical sense of S. Shelah, as well as in the Paris version within the framework of Laskar-Poizat;

5) in the class of *J*-simple theories, a Jonsson version of the Kim-Pillay theorem is obtained for the class of the Jonsson spectrum of an arbitrary model of signature.

The structure and scope of the dissertation. The thesis, with a volume of 84 pages, consists of the following structural elements: designations and abbreviations, introduction, the main three sections, conclusion and list of sources used, containing 81 names.

Summary of the main part of the dissertation. The first section of the dissertation provides preliminary information from model theory, as well as basic concepts and results that reflect the properties of Jonsson theories. In addition, some model-theoretical properties of the class of existentially closed models of such theories are studied.

Subsection 1.1 gives the basic concepts and theorems of the classical theory of models. Subsection 1.2 contains the necessary definitions of concepts and results concerning Jonsson theories.

A theory T is called *Jonsson* if:

1) T has an infinite model;

2) T is inductive (i.e. T is equivalent to set of $\forall\exists$-sentences);

3) T has joint embedding property (*JEP*);

4) T has the amalgamation property (*AP*).

So, for example, Jonsson theories are theories of groups, Abelian groups, Boolean algebras, linear orders, fields of fixed characteristic p, ordered fields.

The *semantic model* C_T of the Jonsson theory T is the ω^+-homogeneous-universal model of the theory T.

We note that for any Jonsson theory a semantic model always exists, therefore, it plays an important role as a semantic invariant.

The semantic model C_T of the Jonsson theory of T is T-existentially closed.

The semantic completion (center) of the Jonsson theory T is the elementary theory T^* of the semantic model C_T of the theory T, that is, $T^* = \mathrm{Th}(C_T)$.

The Jonsson theory T is said to be *perfect* if every semantic model of T is a saturated model of T^*.

Let T be the Jonsson theory, $S^J(X)$ be the set of all existential complete n-types over X that are compatible with T for any finite n, where $X \subset C$. Jonsson theory T is called J-λ-stable if for any T-existentially closed model $\mathfrak{A}$ for any subset X of A, $|X| \le \lambda \Rightarrow |S^J(X)| \le \lambda$.

We have the following results:

Theorem 1.2.9 Let T be a perfect Jonsson theory complete for $\exists$-sentences, $\lambda \ge \omega$. Then the following conditions are equivalent:

(1) T is J-λ-stable;

(2) T^* is λ-stable, where T^* is the center of Jonsson theory T.

Theorem 1.2.11 Let T be a Jonsson theory complete for $\forall\exists$-sentences. Then the following conditions are equivalent:

(1) T is ω-categorical;

(2) T^* is ω-categorical.

In subsection 1.3, we study model-theoretic properties of a class of existentially closed models of Jonsson theories. The following useful statements are proved:

Theorem 1.3.1 Let L be a first-order language, T is a Jonsson theory in the language L, and $\mathfrak{A}$, $\mathfrak{B}$ be existentially closed models of the theory T. Then every $\forall\exists$-sentence of the language L that is true in $\mathfrak{A}$ is also true in $\mathfrak{B}$.

Lemma 1.3.3 Let T be a Jonsson theory. Then for any model $\mathfrak{A} \in E_T$, the theory $\mathrm{Th}_{\forall\exists}(\mathfrak{A})$ is Jonsson.

Theorem 1.3.3 Let T be the Jonsson theory in the language L and $\mathfrak{A}$ be some model of the theory T. Then there exists an existentially closed model $\mathfrak{B}$ of the theory T such that $\mathfrak{A} \subseteq \mathfrak{B}$.

Subsection 1.4 considers the concept of the Jonsson spectrum of an arbitrary model of signature with respect to the concept of cosemanticness, which is a generalization of elementary equivalence in the class of inductive, generally speaking, incomplete theories. The main result of this subsection is the following theorem:

Theorem 1.4.2 Let $[T] \in FrPJSp(\mathfrak{A})/_{\bowtie}$ be an $\exists$-complete class, $\lambda \geq \omega$. Then the following conditions are equivalent:

1) class $[T]$ is ∇-λ- stable;

2) the theory $[T]^*$ is λ-stable, where $[T]^*$ is the center of the class $[T]$.

In Section 1.5, we study the properties of lattices of existential formulas of Δ-Jonsson theories and their relation to well-known issues in model theory. A number of results have been obtained that establish a connection between the properties of the Δ-Jonsson theory, of the central completion of this Δ-Jonsson theory and the properties of the lattice of classes of equivalence of existential formulas with respect to this theory.

In terms of the lattice of existential formulas , the necessary and sufficient conditions for the elimination of quantifiers of central completion of the Jonsson theory T and the positive model completeness of central completion of the Jonsson theory of T are found, and the necessary and sufficient conditions of the perfectness of the Jonsson theory T and the Jonssonness of the center of the Jonsson theory are found.

In terms of the lattice of existential formulas $E_n(T)$ the necessary and sufficient conditions for the elimination of quantifiers of central completion of the Δ-Jonsson theory T and the positive model completeness of central completion of the Δ-Jonsson theory T are found, and the necessary and sufficient conditions of the perfectness of the Δ-Jonsson theory T and the Jonssonness of the center of the Δ-Jonsson theory are found.

Section 2 consider model-theoretical questions of classifying the theory of Abelian groups and the theory of modules with respect to the concept of cosemanticness in the class of Jonsson theories.

Subsection 2.1 contains the necessary elementary information about Abelian groups and their model-theoretic properties.

Subsection 2.2 considered Jonsson analogues of some model-theoretic results for Abelian groups. Namely, the Jonsson Schröder-Bernstein property and the cosemanticness property for Abelian groups are considered.

The following proposition is proved:

Proposition 2.2.1 The theory T_{AG} is the perfect Jonsson theory.

The notion of property SB has been redefined for Jonsson theories and is denoted as JSB, namely: Jonsson theory T has JSB property if for any two existentially closed models $\mathfrak{A}$ and $\mathfrak{B}$ of theory T from the fact that they are mutually isomorphically embedded into each other it follows that they are isomorphic.

The Jonsson analog of J. Goodrick's theorem is obtained, namely:

Theorem 2.2.3 Let T be the Jonsson theory of Abelian groups, then the following conditions are equivalent:

1) T is J-ω-stable;

2) T^* is ω-stable;

3) T has the JSB property.

Also in this subsection obtained a result that describes the semantic model of the Jonsson theory of Abelian groups.

Theorem 2.2.4 Let T be Jonsson theory of Abelian group, then its center $C_T \in E_T$, while C_T is a divisible group and its standard Szmielew group is representable as $\oplus_p Z_{p^\infty}^{(\alpha_p)} \oplus Q^{(\beta)}$, where $\alpha_p, \beta \in \omega^+$, $2^\omega = |C_T|$.

The main result of Subsection 2.2 is the cosemanticity criterion for Abelian groups, which is a Jonsson analogue of the well-known theorem of W. Szmielew on the elementary classification of Abelian groups:

Theorem 2.2.5 If A and B is Abelian groups, then the following conditions are equivalent:

(1) $A \underset{JSp}{\bowtie} B$;

(2) $JInv(JSp(A)/_{\bowtie}) = JInv(JSp(B)/_{\bowtie})$.

Thus, to obtain the cosemanticness of Abelian groups, it is sufficient to compare two invariants of W. Szmielew, namely, invariants of the divisible part.

In Subsection 2.3, we study the Jonsson pairs of the theory of abelian groups in an enriched language. The signature has been extended on one unary predicate. Elements implementing this predicate form an existentially closed submodel of a some model of the considered Jonsson theory. An analogue of W. Szmielew's theorem on the elementary classification of Abelian groups is obtained (Theorem 2.3.3), as well as an analogue of the Schröder-Bernstein property for Jonsson pairs of the theory of Abelian groups (Theorem 2.3.1). The results obtained show a close relationship between the model-theoretic properties of the Jonsson pair and the model-theoretical properties of the center of the considered Jonsson theory.

In subsection 2.4, model-theoretical properties of modules are studied. The following results were obtained in this subsection:

Proposition 2.4.1 Theory T_R is Jonsson theory.

Theorem 2.4.6 Let M_1 and M_2 be two arbitrary R-modules, then the following conditions are equivalent:

(1) $M_1 \underset{JSp}{\bowtie} M_2$;

(2) $JInv(M_1) = JInv(M_2)$.

Theorem 2.4.7 Let T_R be a $\forall\exists$-complete Jonsson theory. Then, if T_R is κ-categorical, where $\kappa \geq \omega$, then T_R is perfect.

Theorem 2.4.9 Let T_R be a theory of R-modules, complete for $\forall\exists$-sentences, $M \in E_{T_R}$. Then the following conditions are equivalent:

(1) $\mathrm{Th}_{\forall\exists}(M)$ – ω- categorical;

(2) $\mathrm{Th}^*_{\forall\exists}(M)$ – ω- categorical, where $\mathrm{Th}^*_{\forall\exists}(M)$ is the center of the theory $\mathrm{Th}_{\forall\exists}(M)$;

(3) for every countable coherent ring R and every countable R-module $A \in E_{T_R}$ exists $n \in \omega$, finite R-modules $B_0,\ldots,B_{n-1}$ and cardinals $\kappa_0,\ldots,\kappa_{n-1} \leq \omega$ such that $A = \bigoplus_{i<n} B_i^{(\kappa_i)}$.

Section 3 is devoted to the study of the abstract property of independence in structures satisfying the Jonsson conditions. In particular, in subsection 3.1, the relations of forking and independence are considered for fragments of Jonsson sets. Axiomatically, the concept of the Jonsson non-forking (*JNF*) is introduced, and the following result is proved with respect to a fragment of some Jonsson set *X*:

Theorem 3.1.1 The following conditions are equivalent:

1) the relation *JNF* satisfies axioms 1–7 with respect to the fragment $Fr(X)$.

2) the theory $Fr(X)^*$ is stable and for any $p \in \mathsf{P}$, $A \in \mathsf{A}$ $(p, A) \in JNF$ $\Leftrightarrow$ p does not fork over A (in the classical meaning of S. Shelah) , where $Fr(X)^*$ is the center of $Fr(X)$.

Also, in subsection 3.1, the relation *JNFLP* (Jonsson non-forking by Lascar-Poizat) is introduced and the following result was obtained:

Theorem 3.1.2 In the *J*-stable existentially complete Jonsson theory, the relation *JNFLP* satisfies axioms 1–7.

The main result of this subsection is the following theorem:

Theorem 3.1.3 If the theory $Fr(X)$ is *J*-stable, then the concepts *JNF* and *JNFLP* are the same.

In subsection 3.2 is considered the concept of Jonsson independence for the Jonsson spectrum of an arbitrary model of signature. In the form of axioms is given relation $J^{\nabla}NF$ (Jonsson ∇-non-forking). For a homogeneous factor spectrum, we have the following result:

Theorem 3.2.2 Let $JSp(A)/_{\bowtie}$ be homogeneous factor spectrum, $[T] \in JSp(A)/_{\bowtie}$ then in class $[T_{\nabla}]$ relation $J^{\nabla}NF$ satisfies axioms 1–7.

Next, we introduce the concept of Jonsson independence between special formula subsets of the semantic model and the concept of a *J*-simple theory. In the class of *J*-simple theories obtained a Jonsson version of the Kim-Pillay theorem, namely:

Theorem 3.2.2 Let the class $[T] \in JSp(A)/_{\bowtie}$ be J - simple, perfect, ∇ - complete. Then for each tuple $\bar{a} \in C_{[T]}$ and ∇ - cl -Jonsson sets $X \subseteq Y$ of the model $C_{[T]}$ the following conditions are equivalent for each $\Delta \in [T]$:

1) $\bar{a} \perp_X^J Y$ in the language of theory Δ;

2) ($tp(\bar{a}/Y), X) \in J^{\nabla}NF$ in the language of theory Δ;

3) for all types $p \in \mathrm{P}$, consistent with $[T]^*$, $X \in \mathrm{X}$ the type p does not fork over X (in the classical meaning of S.Shelah), where $[T]^*$ is the center of the class $[T]$.

The dissertation work was performed at the Department of Algebra, Mathematical Logic and Geometry named after Professor T.G. Mustafin of Buketov Karaganda State University.