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«Well – posedness of a mixed problems of differential equations parabolic type with involution»

Abstract

Of the thesis submitted for the PhD degree in the specialty of
6D060100 – «Mathematics»

In the first section of the thesis, second-order differential operators are considered with the involution of the following types:

1) non-semibounded second-order differential operator with involution

$$-y''(-x) = \lambda y(x), \quad -1 < x < 1,$$

with self-adjoint boundary conditions

$$y(-1) = 0, \quad y(1) = 0,$$

eigenvalues of which are numbers

$$\lambda_{k1} = -(k\pi)^2, \quad k = 0, 1, 2, \dots, \quad \lambda_{k2} = \left(k + \frac{1}{2}\right)^2 \pi^2, \quad k = 0, 1, 2, \dots;$$

and with non-self-adjoint boundary conditions

$$y(-1) = 0, \quad y'(-1) = y'(1),$$

eigenvalues of which are numbers $\lambda_{k1} = -k^2 \pi^2, \lambda_{k2} = k^2 \pi^2$;

2) differential operators of second order with involution with variable coefficient

$$-y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1,$$

with self-adjoint boundary conditions

$$y(-1) = 0, \quad y(1) = 0;$$

and with non-self-adjoint boundary conditions $y(-1) = 0, \quad y'(-1) = y'(1)$;

3) differential operators of second order with involution

$$-y''(x) + \alpha y''(-x) = \lambda y(x), \quad -1 < x < 1,$$

with boundary conditions type

$$y(-1) = 0, \quad y(1) = 0,$$

eigenvalues of which are numbers

$$\lambda_{k_1} = (1 - \alpha) \left(k + \frac{1}{2} \right)^2 \pi^2, \quad k = 0, \pm 1, \pm 2, \dots; \quad \lambda_{k_2} = (1 + \alpha) k^2 \pi^2, \quad k = \pm 1, \pm 2, \dots$$

When $-1 < \alpha < 1$ the operator is semi-bounded. When $\alpha < -1$, $\alpha > 1$ the operator is not semibounded;

4) differential operators of second order with involution

$$-y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1,$$

with boundary conditions type

$$y(-1) = 0, \quad y(1) = 0.$$

The following results are obtained.

1) The Green's function is constructed for a second-order non-semi-bounded differential operator with involution

$$-y''(-x) = \lambda y(x), \quad -1 < x < 1,$$

with self-adjoint boundary conditions

$$y(-1) = 0, \quad y(1) = 0.$$

Получены равномерные оценки функции Грина.

Доказана полнота и базисность Рисса в пространстве $L_2(-1,1)$ собственных функций дифференциального оператора второго порядка с инволюцией
The uniform estimates of the Green function are obtained.
The completeness and basic Riesz in the space $L_2(-1,1)$ of eigenfunctions of a differential operator of second order with involution

$$-y''(-x) = \lambda y(x), \quad -1 < x < 1,$$

with non-self-adjoint boundary conditions is proved.

$$y(-1)=0, \quad y'(-1)=y'(1).$$

The Green function is constructed, the uniform estimates of the Green function are obtained.

2) A theorem on the equiconvergence of expansion of a function $f(x) \in L_1(-1,1)$ in a complete orthonormal system of eigenfunctions of the previous operator with expansion in eigenfunctions of a second-order differential operator with involution with variable coefficient

$$-y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1, \quad q(x) \in L_1(-1,1)$$

with self-adjoint boundary conditions is proved.

$$y(-1)=0, \quad y(1)=0.$$

The basis property is established in the space $L_2(-1,1)$ of eigenfunctions of a second-order differential operator with an involution with variable coefficient.

A theorem on the equiconvergence of the expansion of a function $f(x) \in L_1(-1,1)$ in the system of eigenfunctions of the operator

$$-y''(-x) = \lambda y(x), \quad -1 < x < 1, \quad y(-1)=0, \quad y'(-1)=y'(1)$$

with expansion in eigenfunctions of a differential operator of second order with involution with variable coefficient

$$-y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1, \quad q(x) \in L_1(-1,1)$$

boundary conditions is proved.

$$y(-1)=0, \quad y'(-1)=y'(1).$$

Установлена базисность в пространстве $L_2(-1,1)$ собственных функций дифференциального оператора второго порядка с инволюцией с переменным коэффициентом.

3) Построена функция Грина и получены равномерные оценки функции Грина краевой задачи при $-1 < \alpha < 1$

The basis property is established in the space $L_2(-1,1)$ of eigenfunctions of a second-order differential operator with an involution with variable coefficient.

3) The Green function is constructed and the uniform estimates of the Green function of the boundary value problem are obtained for $-1 < \alpha < 1$

$$-y''(x) + \alpha y''(-x) = \lambda y(x), \quad -1 < x < 1, \quad -1 < \alpha < 1,$$

with boundary conditions type

$$y(-1) = 0, \quad y(1) = 0.$$

4) A theorem on the equiconvergence of expansion of a function $f(x) \in L_1(-1,1)$ in a complete orthonormal system of eigenfunctions of the previous operator with expansion in eigenfunctions of a second-order differential operator with involution with variable coefficient

$$-y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1, \quad q(x) \in L_1(-1,1)$$

with self-adjoint boundary conditions is proved

$$y(-1) = 0, \quad y(1) = 0.$$

Not only the basis property is established, but also the unconditional basis property is proved in the space $L_2(-1,1)$ of eigenfunctions of a second-order differential operator with involution with variable coefficient.

The results of the first section are used in the second section of the dissertation. The results of the second section.

5) A uniform convergence of expansions of functions from a class $L_2(-1,1)$ with respect to eigenfunctions of a boundary value problem is established.

$$-y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x), \quad -1 < x < 1,$$

$$y(-1) = 0, \quad y(1) = 0$$

on a closed gap $-1 < x < 1$. The conditions for the positivity of the eigenvalues of the boundary value problem are established.

6) The Fourier method has proved the existence and uniqueness of the solution of a mixed problem for a parabolic equation with an involution

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - \alpha \frac{\partial^2 u(-x,t)}{\partial x^2} - q(x)u(x,t), \quad -1 < x < 1, \quad t > 0,$$

$$u(x,0) = \varphi(x), \quad u(-1,t) = 0, \quad u(1,t) = 0,$$

where $-1 < \alpha < 1$, with real continuous coefficient $q(x)$ at $-1 < x < 1$.

7) Solvability conditions are found in terms of the initial function of the following ill-posed problems for a parabolic equation with an involution.

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(-x,t)}{\partial x^2}, \quad -1 < x < 1, \quad t > 0, \quad u(x,0) = \varphi(x),$$

with self-adjoint boundary conditions

$$u(-1,t) = 0, \quad u(1,t) = 0,$$

with non- self-adjoint boundary conditions

$$u(-1,t) = 0, \quad u_x(-1,t) = u_x(1,t).$$

The existence and uniqueness of the solution of ill-posed problems is proved, when the initial function is even. The existence and uniqueness of the solution of ill-posed problems is proved, when the initial function is a polynomial in eigenfunctions of the corresponding spectral problem. In this case density is found everywhere in the space $L_2(-1,1)$ of the solvability class of ill-posed problems.

8) Solvability conditions are obtained in terms of the initial function of the following ill-posed problems for a parabolic equation with an involution

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(-x,t)}{\partial x^2} + q(x)u(x,t), \quad -1 < x < 1, \quad t > 0, \quad u(x,0) = \varphi(x),$$

with self-adjoint boundary conditions

$$u(-1,t) = 0, \quad u(1,t) = 0,$$

with non- self-adjoint boundary conditions.

$$u(-1,t) = 0, \quad u_x(-1,t) = u_x(1,t).$$

The existence and uniqueness of the solution of ill-posed problems is proved, when the initial function is a polynomial in eigenfunctions of the corresponding spectral problem. Installed everywhere density in the space $L_2(-1,1)$ of the solvability class of ill-posed problems.

The third section examines the solvability of inverse problems.

9) For the equation of the form of heat conduction with involution

$$u_t(x, t) - u_{xx}(-x, t) = f(x), \quad (x, t) \in \Omega,$$

in a rectangular area $\Omega = \{-\pi < x < \pi, 0 < t < T\}$, solvability questions for the following inverse problem are investigated:

find a pair of functions $u(x, t)$ и $f(x)$ in the region Ω satisfying this equation, conditions

$$u(x, 0) = \varphi(x), u(x, T) = \psi(x), \quad x \in [-\pi, \pi],$$

and homogeneous Dirichlet boundary conditions

$$u(-\pi, t) = 0, u(\pi, t) = 0, \quad t \in [0, T],$$

where $\varphi(x)$ и $\psi(x)$ are given, fairly smooth functions.

The existence and uniqueness of the solution of the inverse problem is proved by Fourier method. The solutions are found in the form of a Fourier series in eigenfunctions of the corresponding spectral problem. A comparative analysis of the solutions of the previous direct problem and this inverse problem is carried out. The conditions for the coincidence of the solutions of the direct problem and the inverse problem with the zero right-hand side are discussed.

10) The inverse problem is investigated as follows. Find the right side $f(x)$ and solution $\Phi(x, t)$ of the equation

$$t^{-\beta} D_t^\alpha \Phi(x, t) - \Phi_{xx}(x, t) - \varepsilon \Phi_{xx}(-x, t) = f(x)$$

in the area of $\Omega = \{(x, t) : -\pi < x < \pi, 0 < t < T\}$ with initial and final conditions of the form

$$\Phi(x, 0) = \phi(x), \Phi(x, T) = \psi(x), \quad x \in [-\pi, \pi],$$

as well as boundary conditions

$$\Phi(-\pi, t) = \Phi(\pi, t), \quad t \in [0, T], \quad \Phi_x(-\pi, t) - \Phi_x(\pi, t) - a\Phi(-\pi, t) = 0.$$

Here the derivative D_t^α is defined as a derivative of order α in the sense of Caputo. Proved the existence and uniqueness of the given problem.