

Annotation of the thesis for the degree of Philosophy Doctor (PhD) in the specialty  
6D060100 - Mathematics

**Structure and scope of the dissertation.** The dissertation consists of an introduction, three sections (each section consists of 3 subsections), a conclusion and a list of references.

**Number of illustrations, tables, used references.** The list of references consists of 75 titles.

**Keywords.** Singular difference operator, unlimited and fluctuating coefficients, weighted normalized spaces, maximal regularity, coercive estimate, compact resolvent.

**Relevance of the topic.** In the theory of unbounded operators for differential and difference operators, the main tasks are to obtain conditions for their reversibility and regularity, as well as to obtain spectral and approximate properties of the resolvent.

Ordinary high-order differential operators defined in the numerical axis, in connection with the problems of quantum mechanics began to be studied in the first half of the twentieth century. In the case when the operator is self-adjoint and its intermediate coefficients are almost constant, or their growth is bounded above by a certain degree of potential, the above problems are widely studied. The results can be found in the famous monographs of M.A. Naimark, M.V. Fedoryuk, B.M. Levitan and I.S. Sargsyan.

The objects of application of differential operators and their difference analogues correspond to different modes of the same process, despite this, the development of the theory of operators corresponding to infinite systems of difference equations (from now on we will call them the infinite difference operator) lags significantly behind the theory of singular differential operators. There are significant differences between them. For example, the Cauchy problem for a differential equation is first solved in small interval, where the independent variable changes, and in the case of the Cauchy problem for a difference equation, unfortunately, the concept of a “small interval” does not exist. Secondly, the some results of our study show that the regularity conditions for difference operators are much weaker than the regularity conditions for their differential analogues. Similar facts are found in practical problems. From the foregoing, it follows the need for a separate study of difference operators using new methods.

Infinite difference operators of the Sturm-Liouville type were investigated in the works of M. Otelbaev, B. Muslimov, R.P. Agarval, S. Chevas, C. Lisama, A. Avila and S. Jitomirskaya (A. Avila - 2014 Fields Prize winner), and weighted Sobolev difference spaces, associated with them, were investigated in the works of E.S. Smailov, A.T. Bulabaev and L.M. Mustafina.

Recently, in connection with applications in stochastic processes of modeling of the problems related to the dynamics of Brownian motion, in problems of modeling vibrations in a medium with resistance and in a compressible medium, in problems of

biology and financial mathematics, the studies of second-order differential and difference operators with independently growing intermediate coefficients have grown sharply. Among them, the case of operators with intermediate coefficients, which is markedly different from Sturm-Liouville type operators, is of particular interest to specialists. Second-order differential operators whose intermediate coefficients increase in order close to linear were considered by A. Lunardi, V. Vespri, G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt, M. Hieber, L. Lorenzi. And the question of studying the case of an infinite second-order difference operator, except for some works by K.N. Ospanov, R.D. Akhmetkalieva and A. Zulkhazhav, remains open.

Therefore, the questions of correctness, regularity, and spectrum for the second and higher orders difference and differential operators with unbounded intermediate coefficients are not fully studied and, accordingly, are relevant.

**Purpose of the study.** Obtain the conditions for the invertibility and coerciveness of difference and differential operators with unbounded intermediate coefficients of the second and higher orders and show some of their spectral properties.

**Object of study.** The questions of the correctness and separability of infinite difference and differential operators of the second and higher orders having unbounded intermediate coefficients.

**Research methods.** The work uses methods of a priori estimates, construction of pseudoresolvents using local problems, the separability theory of operators, weighted Hardy-type inequalities, and well-known perturbation theorems.

**Scientific novelty and the practical value of the work.** In the work, new types of the second and higher orders difference and differential operators with intermediate coefficients whose growth is independent of the potential and can fluctuate are studied, and conditions for their correctness and separability, as well as a sufficient condition for the discreteness of the spectrum of a degenerate operator, are obtained. Since in the indicated case the methods of the theory of well-known Schrödinger operators are not applicable, a new technique has been developed based on the use of Hardy weighted inequalities and Friedrichs functionals, as well as the apparatus for averaging the intermediate coefficient.

As a result, the continuous invertibility and separability of some classes of difference and differential operators with priority growing intermediate coefficients is proved, although the latter are not strictly positive. It is also shown that in coercive estimates for difference operators, the constants depend on the step of the difference, they are inversely proportional to some degree of it (its order is indicated).

The results of the dissertation are theoretical. They can be used to study the qualitative properties of singular operators. They are also useful in modeling of stochastic processes.

**Provisions for protection.** To the defense are taken out:

1<sup>0</sup> Hardy-type difference inequality, where the weighted norm of a numerical sequence is estimated through the weighted norm of its high-order difference;

2<sup>0</sup> Sufficient conditions for invertibility in a Hilbert space of an infinite high even order difference operator with priority growing intermediate coefficients;

3<sup>0</sup> Description of the domain of above operator and a coercive estimate of its elements independent of coefficient fluctuations;

4<sup>0</sup> The separability of one degenerate infinite second-order difference operator whose intermediate coefficient is not separated from zero and rapidly oscillates, as well as the discreteness of its spectrum;

5<sup>0</sup> Separability conditions in a Hilbert space of fourth-order differential operator with oscillating coefficients and second-order differential operator with complex-valued coefficients.

**Publications.** The main results of the dissertation are published in 9 papers. Of these, 3 articles in publications recommended by KKSON MES RK, 1 article in a journal from the Scopus list with non-zero IF, 1 article in a foreign publication, 4 works in materials of international scientific conferences, including 1 in the proceedings of a foreign conference.

### Summary of work

In subsection 1.1 of the first section, the main definitions are given and, in part, with the proofs, auxiliary statements are necessary to describe the main results of the dissertation.

Subsection 1.2 considers the difference operator

$$L_0 y = h^{-2} \Delta^{(2)} y + h^{-1} r \Delta_+ y + h^{-1} \overline{s \Delta_+ y} + q y + p \bar{y},$$

where  $y = \{y_{jh}\}_{j=-\infty}^{+\infty}$ ,  $\bar{y} = \{\bar{y}_{jh}\}_{j=-\infty}^{+\infty}$ ,  $L_0 y = \{(L_0 y)_{jh}\}_{j=-\infty}^{+\infty}$ ,  $f = \{f_{jh}\}_{j=-\infty}^{+\infty}$ ,  $\Delta_+ y = \{(\Delta_+ y)_{jh}\}_{j=-\infty}^{+\infty}$ ,  $\overline{\Delta_+ y} = \{\overline{(\Delta_+ y)_{jh}}\}_{j=-\infty}^{+\infty}$ ,  $\Delta^{(2)} y = \{(\Delta^{(2)} y)_{jh}\}_{j=-\infty}^{+\infty}$ , and  $r = \text{diag}\{r_{jh}, j \in Z\}$ ,  $s = \text{diag}\{s_{jh}, j \in Z\}$ ,  $q = \text{diag}\{q_{jh}, j \in Z\}$ ,  $p = \text{diag}\{p_{jh}, j \in Z\}$  are diagonal matrices,  $h > 0$ ,  $r_{jh}$  are given real numbers,  $s_{jh}$ ,  $q_{jh}$ ,  $p_{jh}$ ,  $f_{jh}$  are complex numbers,  $\bar{y}_{jh}$  is a complex conjugate of  $y_{jh}$ , and  $\Delta_+ y_{jh} = y_{(j+1)h} - y_{jh}$ ,  $\overline{\Delta_+ y_{jh}} = \overline{y_{(j+1)h} - y_{jh}}$ ,  $\Delta^{(2)} y_{jh} = y_{(j+1)h} - 2y_{jh} + y_{(j-1)h}$  ( $j \in Z$ ). We assume that,  $f \in l_2(h)$ ,  $l_2(h)$  is the space of elements  $y = \{y_{jh}\}_{j=-\infty}^{+\infty}$  with norm

$\|y\|_{2,h} = \left( \sum_{j=-\infty}^{+\infty} |y_{jh}|^2 h \right)^{1/2}$ . Let the operator  $L_0$  be defined in the set of compactly supported sequences

$\tilde{l} = \left\{ y = \{y_{jh}\}_{j=-\infty}^{+\infty} : \exists M > 0, y_{jh} = 0 \ \forall j \geq M \right\}$ . By  $L$  we denote the closure of the operator  $L_0$  in  $l_2(h)$ . We introduce the following notation:

$$\alpha_{\varphi,\psi}(n) = \left( \sum_{j=0}^n |\varphi_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=n}^{+\infty} \psi_j^{-2} \right)^{\frac{1}{2}} \quad (n = 0, 1, 2, \dots),$$

$$\beta_{\varphi,\psi}(k) = \left[ \left( \sum_{j=k}^{-1} |\varphi_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=-\infty}^k \psi_j^{-2} \right)^{\frac{1}{2}} \right] \quad (k = -1, -2, \dots),$$

$$\gamma_{\varphi,\psi} = \max \left( \sup_{n=0,1,2,\dots} \alpha_{\varphi,\psi}(n), \sup_{k=-1,-2,\dots} \beta_{\varphi,\psi}(k) \right),$$

where  $\phi = \text{diag} \{ \phi_j, j \in Z \}$  and  $\psi = \text{diag} \{ \psi_j, j \in Z \}$  ( $\psi_j \neq 0$ ,  $\psi_j$  are real numbers) are given diagonal matrices.

**Theorem 0.1** Let the matrix  $r$  satisfy the conditions  $r_{jh} \geq 1$  ( $j \in Z$ ),

$$\sup_{n=0,1,2,\dots} \alpha_{1,r}(n) < \infty, \quad (1)$$

and

$$\sup_{k=-1,-2,\dots} \beta_{1,r}(k) < \infty. \quad (2)$$

And for the matrices  $s$ ,  $q$  and  $p$  the following conditions are satisfied:

$$r_{jh} \geq 8[1 + 2\gamma_{1,r}] \left( 1 + \frac{2}{h} + \sqrt{\frac{1}{h} \left( 1 + \frac{2}{h} \right)} \right) |s_{jh}| + \delta \quad (j \in Z), \quad \delta > 0,$$

$$\gamma_{q,r} < \infty, \quad \gamma_{p,r} < \infty.$$

Then the operator  $L$  is invertible, and the inverse operator  $L^{-1}$  is defined in the whole space  $l_2(h)$ , and for each  $y \in D(L)$  the estimate

$$\|h^{-2}\Delta^{(2)}y\|_2 + \|h^{-1}r\Delta_+y\|_2 + \|h^{-1}s\overline{\Delta_+y}\|_2 + \|qy\|_2 + \|p\overline{y}\|_2 \leq C_1(h)\|L_0y\|_2 \quad (3)$$

holds, where the constant  $C_1$  in a neighborhood of  $h=0$  satisfies the conditions

$$\frac{T_1}{h} \leq C_1 \leq \frac{T_2}{h} \quad (T_1, T_2 \text{ are constants}).$$

**Definition 0.1** It is said that the operator  $L$  is separable in the space  $l_2(h)$  if inequality (3) holds for  $y \in D(L)$ . Inequality (3) itself is called a coercive estimate.

The operator  $L_0$  is not symmetric. In the particular case when,  $s = p = 0$ ,  $q = \bar{q}$ , the separability conditions for the operator  $L_0$  were obtained by K.N. Ospanov and A. Zulkhazhav<sup>1</sup>. This result follows from Theorem 0.1 in the particular case. In addition, in Theorem 0.1 there is no condition on the fluctuation of the coefficient  $r$ , and weaker conditions (1), (2) are given.

In subsection 1.3 of the work, we consider the second-order difference operator

$$(m_0y)_i = \Delta^{(2)}y_i + (r\Delta_+y)_i + (qy)_i, \quad i \in Z,$$

with an oscillating coefficient and sufficient conditions are obtained for its continuous invertibility and separability. We will use the notation from subsection 1.2. And let

$$y = \{y_j\}_{j=-\infty}^{+\infty}, \quad \Delta^{(2)}y = \Delta_-(\Delta_+y), \quad m_0y = \{(m_0y)_j\}_{j=-\infty}^{+\infty}, \quad \Delta_+y = \{(\Delta_+y)_j\}_{j=-\infty}^{+\infty},$$

$\Delta^{(2)}y = \{\Delta^{(2)}y_j\}_{j=-\infty}^{+\infty}$ , and  $r = \text{diag} \{ r_j, j \in Z \}$ ,  $q = \text{diag} \{ q_j, j \in Z \}$  be real diagonal matrices. Let  $m$  denote the closure in  $l_2$  of the operator  $m_0y = \Delta^{(2)}y + r\Delta_+y + qy$  defined on the set  $\tilde{l}$ .

<sup>1</sup>Ospanov K., Zulkhazhav A. On the properties of solutions of one system of second order difference equations//Bulletin of the Karaganda University. Mathematics Serial. – 2015. –T.78, №2.

Let  $r_j \geq 0$  ( $j \in \mathbb{Z}$ ), and numbers  $n_j$  ( $j \in \mathbb{Z}$ ) be chosen as follows

$$n_j = \begin{cases} \max \left\{ k \geq 0 : \frac{1}{(1+k)} \geq \sum_{i=j-k}^{j+k} r_i^2 \right\}, & r_j < 1, \\ 0, & r_j \geq 1, \end{cases}$$

and through them we determine the sequence  $\{B_j\}_{j=-\infty}^{+\infty}$  as follows:

$$B_j = \begin{cases} 2(n_j + 1), & r_j < 1, \\ r_j^{-1}, & r_j \geq 1. \end{cases} \quad (4)$$

We introduce the following notation:

$$B_+ = \text{diag} \{ B_i, i = 0, 1, 2, \dots \}, \quad B_- = \text{diag} \{ B_j, j = -1, -2, \dots \},$$

$$\gamma_{q, B_+} = \sup_{m \geq 0} \left[ \left\{ \sum_{i=0}^m q_i^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=m}^{+\infty} B_i^2 \right\}^{\frac{1}{2}} \right] < \infty, \quad \gamma'_{q, B_-} = \sup_{\tau < 0} \left[ \left\{ \sum_{i=\tau}^{-1} q_i^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=-\infty}^{\tau} B_i^2 \right\}^{\frac{1}{2}} \right] < \infty,$$

where  $q = \text{diag} \{ q_j, j \in \mathbb{Z} \}$  and  $B_i$  ( $i \in \mathbb{Z}$ ) are defined by equalities (4).

**Theorem 0.2** Let real diagonal matrices  $r, q$  satisfy the conditions

$$r_j \geq 0 \quad (j \in \mathbb{Z}), \quad (5)$$

$$\gamma''_{|q|+E, B_+, B_-} = \max(\gamma_{|q|+E, B_+}, \gamma'_{|q|+E, B_-}) < \infty,$$

where  $|q|+E = \text{diag} \{ |q_j|+1, j \in \mathbb{Z} \}$  ( $E$  is the identity matrix). Then the operator  $m$  is continuously invertible in the space  $l_2$  and separable, i.e. for each  $y \in D(m)$ , the inequality

$$\|\Delta^{(2)}y\|_2 + \|r\Delta_+y\|_2 + \|(|q|+E)y\|_2 \leq C\|my\|_2$$

holds.

Note that the operator  $m_0$  is a special case of  $L_0$  considered in Subsection 1.2. However, in the above theorem, the condition  $r_j \geq 1$  from Theorem 0.1 is replaced by the weaker condition (5), moreover, the conditions of Theorem 0.2 are satisfied for the matrix  $r$  with rapidly oscillating elements.

**Example 0.1** The conditions of Theorem 0.2 are satisfied for the following minimal closed operator

$$ly = \Delta^{(2)}y + p\Delta_+y + y$$

with  $p = \text{diag} \{ |j|^\alpha, j \in \mathbb{Z} \}$ ,  $\alpha > 1$ . Therefore, the operator  $l$  in  $l_2$  is continuously invertible and the following inequality holds for each  $y \in D(l)$ :

$$\|\Delta^{(2)}y\|_2 + \left\{ \sum_{j=-\infty}^{+\infty} |j|^{2\alpha} |\Delta_+y_j|^2 \right\}^{1/2} + \|y\|_2 \leq C\|ly\|_2.$$

The following theorem establishes sufficient conditions for the discreteness of the spectrum of  $m$ . This statement is important, for example, for solving the infinite difference equation

$$my = \Delta^{(2)}y + r\Delta_+y + qy = f.$$

**Theorem 0.3** Let the conditions  $\gamma''_{q,B_-,B_+} = \max(\gamma_{q,B_+}, \gamma'_{q,B_-}) < \infty$  and

$$\lim_{m \rightarrow +\infty} m^{\frac{1}{2}} \left( \sum_{i=m}^{+\infty} B_i^2 \right)^{\frac{1}{2}} + \lim_{k \rightarrow -\infty} (|k|)^{\frac{1}{2}} \left( \sum_{i=-\infty}^k B_i^2 \right)^{\frac{1}{2}} = 0$$

be satisfied for  $r = \text{diag} \{ r_j, j \in \mathbb{Z} \}$   $r_i \geq 0$  ( $i \in \mathbb{Z}$ ) and  $q = \text{diag} \{ q_j, j \in \mathbb{Z} \}$ . Then the operator  $m^{-1}$  is compact in the space  $l_2$ .

In the second part of the thesis, a difference operator of high even order with an intermediate coefficient is considered. In subsection 2.1, in order to improve the research apparatus, one new difference Hardy type inequality is proved. Let  $s$  be a natural number and  $\Delta^{(2s)} y = \Delta^{(2)} \Delta^{(2s-2)} y$ ,  $\Delta^{(2s-1)} y = \Delta_+ \underbrace{\Delta^{(2)} \Delta^{(2)} \dots \Delta^{(2)}}_{s-1} y$ . We denote by  $\hat{H}_{p,v}^{(k)}$

the space with norm

$$\|a\|_{\hat{H}_{p,v}^{(k)}} = \left( \sum_{s=-\infty}^{+\infty} |v_s \Delta^{(k)} a_s|^p \right)^{\frac{1}{p}} \quad (a = \{a_s\}_{s=-\infty}^{+\infty}).$$

For the numerical matrices  $u = \text{diag} \{ u_j, j \in \mathbb{Z} \}$  and  $v = \text{diag} \{ v_j, j \in \mathbb{Z} \}$ , we introduce the following notation:

$$\begin{aligned} T_{m,u,v} &= \sup_{n=0,1,2,\dots} \left( \sum_{j=0}^n |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=n}^{+\infty} j^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}}, \\ T''_{m,u,v} &= \sup_{\tau < 0} \left( \sum_{j=\tau}^0 |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=-\infty}^{\tau} |j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}}, \\ T_{0,m,u,v} &= \sup_{n=0,1,2,\dots} \left( \sum_{j=0}^n |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=n}^{+\infty} ((j-n)^{(m-1)})^{p'} |v_j|^{-p'} \right)^{\frac{1}{p'}}, \\ T''_{0,m,u,v} &= \sup_{\tau=0,-1,-2,\dots} \left( \sum_{j=\tau}^0 |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=-\infty}^{\tau} |\tau-j|^{(m-1)p'} |v_j|^{-p'} \right)^{\frac{1}{p'}}, \end{aligned}$$

where  $p' = p/(p-1)$  and  $m$  is a natural number.

**Theorem 0.4** Let  $1 < p < \infty$ ,  $m \geq 2$ , and the numerical matrices  $u = \text{diag} \{ u_j, j \in \mathbb{Z} \}$  and  $v = \text{diag} \{ v_j, j \in \mathbb{Z} \}$  ( $v_j \neq 0 \forall j \in \mathbb{Z}$ ) satisfy condition  $\gamma_{m,u,v} = \sqrt[p]{\max \left[ (T_{m,u,v})^p, (T''_{m,u,v})^p \right]} < +\infty$ .

Then for each  $y = \{y_j\}_{j=-\infty}^{+\infty} \in \tilde{l}$  the inequality

$$\sum_{j=-\infty}^{+\infty} |u_j y_j|^p \leq C_{2,m,u,v}^p \sum_{j=-\infty}^{+\infty} |v_j \Delta^{(m)} y_j|^p \quad (6)$$

holds. In addition, if  $C_{2,m,u,v}$  is a minimal constant satisfying (6), then

$$\gamma'_{m,u,v} = \sqrt[p]{\min \left[ (A_+ T_{0,m,u,v})^p, (A_- T''_{0,m,u,v})^p \right]} \leq C_{2,m,u,v} \leq p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \gamma_{m,u,v},$$

where  $p' = p/(p-1)$ , and  $A_+$ ,  $A_-$  are constants.

In the case  $m=1$ , some difference Hardy-type inequalities were proved in the

works of G. Mukhamediev, K.F. Andersen and H.P. Heinig<sup>2</sup>.

In subsection 2.3, we consider the following difference operator:

$$\tilde{L}_0 y = \Delta^{(2n)} y + r \Delta^{(2n-1)} y + s \overline{\Delta^{(2n-1)} y} + \sum_{j=1}^{2n-1} \left( Q^{(j)} \Delta^{(2n-j-1)} y + P^{(j)} \overline{\Delta^{(2n-j-1)} y} \right),$$

where  $y = \{y_k\}_{k=-\infty}^{+\infty}$ ,  $\Delta_+ y_k = y_{k+1} - y_k$ ,  $\Delta_- y_k = y_k - y_{k-1}$ ,  $\Delta^{(2)} y_k = \Delta_- \Delta_+ y_k = y_{k+1} - 2y_k + y_{k-1}$  ( $k \in \mathbb{Z}$ ),  $\Delta^{(2s)} y = \Delta^{(2)} \Delta^{(2s-2)} y$ ,  $\Delta^{(2s-1)} y = \Delta_+ \underbrace{\Delta^{(2)} \Delta^{(2)} \dots \Delta^{(2)}}_{s-1} y$  ( $s \in \mathbb{N}$ ), and  $r = \text{diag} \{ r_j, j \in \mathbb{Z} \}$ ,  $s = \text{diag} \{ s_j, j \in \mathbb{Z} \}$ ,  $Q^{(\theta)} = \text{diag} \{ q_j^{(\theta)}, j \in \mathbb{Z} \}$ ,  $P^{(\theta)} = \text{diag} \{ p_j^{(\theta)}, j \in \mathbb{Z} \}$  ( $\theta = \overline{1, 2n-1}$ ) are given diagonal matrices. We denote by  $\tilde{L}$  the closure in the norm of  $l_2$  of the operator  $\tilde{L}_0$  defined on the set  $\tilde{l}$ . The following theorem is the main result of this subsection.

**Theorem 0.5** Let the matrices  $r = \text{diag} \{ r_j, j \in \mathbb{Z} \}$ ,  $s = \text{diag} \{ s_j, j \in \mathbb{Z} \}$ ,  $Q^{(\theta)} = \text{diag} \{ q_j^{(\theta)}, j \in \mathbb{Z} \}$ ,  $P^{(\theta)} = \text{diag} \{ p_j^{(\theta)}, j \in \mathbb{Z} \}$  ( $\theta = \overline{1, 2n-1}$ ) satisfy the following conditions:

$$\max \left( \gamma_{2n-1, E, r}, \gamma_{2n-\theta, P^{(2n-\theta)}, r}, \gamma_{2n-\theta, Q^{(2n-\theta)}, r} \right) < \infty \quad (\theta = \overline{1, 2n-1}),$$

$$|s_j| \leq \alpha_1 r_j \quad (j \in \mathbb{Z}), \quad 0 < \alpha_1 < \frac{1}{5\sqrt{2}},$$

where  $E$  is the identity matrix. Then the difference operator  $\tilde{L}$  is continuously invertible and for each  $y \in D(\tilde{L})$  the following estimate

$$\left\| \Delta^{(2n)} y \right\|_2 + \left\| r \Delta^{(2n-1)} y \right\|_2 + \left\| s \overline{\Delta^{(2n-1)} y} \right\|_2 + \sum_{j=1}^{2n-1} \left( \left\| Q^{(j)} \Delta^{(2n-j-1)} y \right\|_2 + \left\| P^{(j)} \overline{\Delta^{(2n-j-1)} y} \right\|_2 \right) \leq C_1 \left\| \tilde{L} y \right\|_2$$

holds.

In the third section of the dissertation, we give conditions for the continuous invertibility and separability of some second and fourth order differential operators and compare them with the conditions obtained by us in the previous sections for difference operators. Here we took into account the fact that the simulation of real dynamic processes lead mainly to the second and fourth order differential or difference equations.

In subsection 3.1, we obtain conditions for the correctness and separability in the space  $L_2(R)$  of a second-order degenerate differential operator  $ly = -y'' + r(x)y' + q(x)\bar{y}' + s(x)y + p(x)\bar{y}$  with complex-valued coefficients. We assume that  $r$  and  $q$  are continuously differentiable, and  $s$  and  $p$  are continuous, and  $y = y_1 + iy_2$  and  $\bar{y} = y_1 - iy_2$ . The differential expression  $ly$  we first define on the set  $C_0^{(2)}(R)$  of compactly supported and twice continuously differentiable functions, and its closure in space  $L_2(R)$  we denote again by  $l$ .

For given continuous functions  $g$  and  $h \neq 0$ , we denote

<sup>2</sup> Andersen K.F., Heinig H.P. Weighted norm inequalities for certain integral operators // SIAM J. Math. -1983. Vol.14, -P. 834-844.

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \|h^{-1}\|_{L_2(t,+\infty)} \quad (t > 0), \quad \beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|h^{-1}\|_{L_2(-\infty,\tau)} \quad (\tau < 0),$$

$$\gamma_{g,h} = \max(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)).$$

**Theorem 0.6** Let the functions  $r$  and  $q$  be continuously differentiable, and  $s$  and  $p$  be continuous and satisfy the conditions

$$\sqrt{|\operatorname{Re} r|} - \omega(|\operatorname{Im} r| + |q|) \geq 1 \text{ и } (1 < \omega < 2), \quad \gamma_{1+|s|+|p|, \sqrt{|\operatorname{Re} r|}} < \infty.$$

Then the inverse of  $l^{-1}$  to the operator  $l$  exists and is defined on the whole space  $L_2(R)$ .

**Theorem 0.7** Let the functions  $r, q, s$  and  $p$  satisfy conditions of Theorem 0.6 and

$$\sup_{|x-\eta| \leq 1} \frac{\operatorname{Re} r(x)}{\operatorname{Re} r(\eta)} < +\infty.$$

Then for each element  $y$  from the domain  $D(l)$  of the operator  $l$  the following inequality holds:

$$\|y''\|_2 + \|ry'\|_2 + \|q\bar{y}\|_2 + \|sy\|_2 + \|p\bar{y}\|_2 \leq C \|ly\|_2.$$

Theorems 0.6 and 0.7 generalize the results of K.N. Ospanov and R.D. Akhmetkalieva<sup>3</sup>.

In Subsection 3.2, we obtain the conditions for the correctness and separability in  $L_2(R)$  of a fourth-order differential operator  $M_0 y = -y^{(4)} + p(x)y'' + s(x)y' + \theta(x)y$ . Here  $p$  is twice continuously differentiable, and  $\theta$  is a continuous function. The differential operator  $M_0 y$  we define in the set  $C_0^{(4)}(R)$  of compactly supported and four times continuously differentiable functions, and its closure in  $L_2(R)$  we denote by  $M$ . The results of this subsection also occur in some cases where the coefficients fluctuate rapidly. To do this, we introduce the following averaged functions:

$$v^*(x) = \sup \left\{ d : d^{-1} \geq \int_{x-d/2}^{x+d/2} v(t) dt \right\},$$

$$v_n^*(x) = \inf \left\{ d^{-1} : d^{-2n+1} \geq \int_{x-d}^{x+d} v^2(t) dt \right\} \quad (n=1,2), \quad x \in R,$$

where  $v(x)$  is a given non-negative continuous function. We take the functions:

$$v_{g,h,\delta_+}(t) = \left( \int_0^t g^2(\xi) d\xi \right)^{1/2} \left( \int_{t-\delta_+}^{+\infty} \xi h^{-2}(\xi) d\xi \right)^{1/2} \quad (t > 0),$$

$$\mu_{g,h,\delta_-}(\tau) = \left( \int_\tau^0 g^2(\xi) d\xi \right)^{1/2} \left( \int_{-\infty}^{\tau+\delta_-} \xi h^{-2}(\xi) d\xi \right)^{1/2} \quad (\tau < 0)$$

<sup>3</sup> K. Ospanov, R. Akhmetkalieva. Separation and the existence theorem for second order nonlinear differential equation// Elect. J. Qual. Th. Diff. Equ. -2012. –Vol.66. –P.1–12.

and denote

$$\omega_{g,h,\delta_+,\delta_-} = \max \left( \sup_{t>0} v_{g,h,\delta_+}(t), \sup_{\tau<0} \mu_{g,h,\delta_-}(\tau) \right).$$

**Theorem 0.8** Let a function  $p \geq 1$  be twice continuously differentiable and satisfy the following conditions:

- 1)  $\omega_{1,(\sqrt{r})_1,0,0} p < +\infty$ ,
- 2)  $\frac{1}{a} \leq \frac{p^*(x)}{p^*(\eta)} \leq a \quad \forall \eta \in \left( x - \frac{b}{2} p^*(x), x + \frac{b}{2} p^*(x) \right), x \in R$ , where  $a \geq 1, b > 0$  and  $a^3 b^{-1} < \infty$ ;

$$3) A(p, p^*) = \sup_{x \in R} \left[ \left( \int_{x - \frac{p^*(x)}{4}}^{x + \frac{p^*(x)}{4}} p^2(t) dt \right)^{1/2} \cdot (p^*(x))^{\frac{3}{2}} \right] < \infty;$$

- 4) for a continuously differentiable function  $s$  and a continuous function  $\theta$ , there are numbers  $\delta_+ > 0$  and  $\delta_- > 0$  such that  $\rho_{s,\theta} = \max \left[ \omega_{\theta, r_2^*, \delta_+, \delta_-}, \gamma_{s, r_2^*} \right] < +\infty$ .

Then for  $M$  there exists an inverse operator  $M^{-1}$  defined on the whole of  $L_2(R)$ . Moreover, for each  $y \in D(M)$  we have the inequality

$$\|y^{(4)}\|_2 + \|py''\|_2 + \|sy'\|_2 + \|\theta y\|_2 \leq C \|f\|_2.$$

In Theorem 0.8, the reversibility and separability of a fourth-order differential operator with an increasing and rapidly oscillating intermediate coefficient is apparently proved for the first time.

In conclusion, the obtained results of the dissertation are briefly described and some areas of their application are indicated.

Thus, in the dissertation, some classes of linear difference and singular differential operators with rapidly growing intermediate coefficients were investigated for solvability and the following new scientific results were obtained:

- in order to improve the research apparatus, a Hardy-type difference inequality is proved, where the weighted  $l_p$ -norm ( $1 < p < \infty$ ) of a numerical sequence from above is estimated through the weighted  $l_p$ -norm of its high-order difference;
- the conditions of invertibility in the Hilbert space of an infinite difference operator with increasing intermediate coefficients of high even order are obtained, its domain is completely described, its separability is proved, and the corresponding coercive estimate is given;
- the conditions of correctness and separability of a degenerate infinite difference operator with rapidly oscillating intermediate coefficients in a Hilbert space are shown, and the discreteness conditions of its spectrum are found;
- the conditions of continuous invertibility and separability of a second-order differential operator with complex-valued coefficients are shown;

- conditions of correctness and separability in a Hilbert space of one fourth-order differential operator with oscillating coefficients are obtained. These separability conditions were compared with the separability conditions of difference analogs of the operator and it was shown that the conditions in the difference case are much weaker.