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# «CONES GENERATED BY A GENERALIZED FRACTIONAL-MAXIMAL FUNCTION AND EMBEDDINGS IN REARRANGEMENT INVARIANT SPACES»

## ABSTRACT
**of the dissertation work for the degree of Doctor of Philosophy (PhD) in the educational program**
**8D05401 – Mathematics**

**The relevance of the topic.** Classical integral operators, fractional maximal functions and Riesz potential play an important role in harmonic analysis, in theory of function spaces, in theory of potentials and in solving differential equations.

The theory of maximal operators and integrals of potential type in function spaces has a rich history and many works are devoted to it (M.Riesz, G.H. Hardy, D.Littlewood, S.Sobolev, S.M. Nikolsky, O.V. Besov, V.P. Ilyin, H.Triebel, E.Stein, G.Weiss, E.Nakai, P.I.Lizorkin, D.R.Adams, L.I. Hedberg, S.Samko, V.Kokilashvili, V.I. Burenkov, M.L. Goldman, A.Gogatishvili, V.Guliyev, etc.). The problems of the boundedness in Lebesgue space of the classical maximal operator, fractional maximal operator, and the Riesz potential are widely known.

In the dissertation work considered the space of generalized fractional maximal functions and the space of the generalized Riesz potential, based on rearrangement-invariant spaces. These two operators are defined by some function $\Phi$ and the potential kernel $G$, respectively.

In contrast to the classical fractional maximal functions and the classical Riesz potential defined by power functions, are considered the generalized fractional maximal function $M_\Phi$, Defined by function $\Phi$, from suitable classes, and the corresponding the generalized Riesz potential $I_G$ defined by kernel $G$. The generalized fractional maximal functions we considered include more general functions (not necessarily in the form of power functions). Such generalizations provide greater flexibility in describing the differential properties of a function, allow one to obtain new meaningful results and prove theorems in cases where classical potentials and maximal functions do not work.

With the help of such operators, a space of generalized fractional maximal functions modelled on the rearrangement-invariant spaces are defined and the problems of embeddings into rearrangement-invariant spaces are studied. The study such an embeddings leads to various cones formed by monotonically nonincreasing functions. The properties of cones constructed from nonincreasing rearrangements of generalized fractional maximal functions are studied. With the help of such cones, criteria for embedding the space of generalized fractional maximal functions into rearrangement-invariant spaces are formulated. In this case, problems of mutual covering of cones play an important role.

In recent decades, the issues of boundedness of various classical integral operators in Morrey-type spaces have been actively studied (V.I. Burenkov,

V.Guliev, A.Gogatishvili, R.Mustafaev, M.L. Goldman, I.Chen, T.Mizuhara, E.Nakai, W.Yuan, W.Sickel, D.Yang, E.D. Nursultanov, R. Oynarov, N.A. Bokayev, etc.).

The fundamental role of the integral operators in the embedding theorems into rearrangement-invariant spaces, in the theory of the function spaces and their use in the theory of partial differential equations are widely known.

**The goal of the work:**

- define a generalized fractional-maximal function and spaces of generalized fractional-maximal functions, investigate the issues of embedding such spaces in rearrangement-invariant spaces;
- obtain estimates of non-increasing rearrangements of the generalized fractional maximal function;
- consider various cones generated by a non-increasing rearrangement of the generalized fractional maximal function and obtain conditions for their mutual covering.

**Research objectives.** The purpose of this work is to study the following problems: defining a generalized fractional maximal function and considering the space of generalized fractional maximal functions; obtaining various estimates for a non-increasing rearrangement of a generalized fractional maximal function, constructing various cones associated with its non-increasing rearrangement, and obtaining conditions for the mutual covering of such cones; obtaining conditions for embedding the space of generalized fractional maximal functions into rearrangement-invariant spaces, obtaining a description of the optimal rearrangement-invariant space for such an embedding; obtaining conditions for the boundedness of a supremal operator in a weighted Lorentz space.

**Object of the research** is the space of generalized fractional maximal functions and the cones generated by them. Embedding the space of generalized fractional maximal functions into rearrangement-invariant spaces.

**Subject of the research.** The space of generalized fractional maximal functions and its embedding in rearrangement-invariant spaces.

**Research methodology.** The main research methods are methods of the theory of operators in function spaces. Using non-increasing rearrangements of functions, constructing cones formed by non-increasing functions, and conditions for their equivalence. Construction of an optimal rearrangement-invariant space for embedding the space of generalized fractional maximal functions.

To implement such problems, it is necessary to obtain estimates for a non-increasing rearrangement of a generalized fractional maximal function.

**Scientific novelty.** In this work, the following new results are obtained: a generalized fractional-maximal function is defined, which coincides, in a particular case, with the classical fractional-maximal function; various estimates for the non-increasing rearrangement of the generalized fractional maximal function are obtained; various cones associated with a non-increasing rearrangement of a function were constructed and the issues of mutual covering of such cones were investigated; the connections between the generalized fractional maximal function and the generalized Riesz potential are considered; conditions for embedding the

space of generalized fractional maximal functions into rearrangement-invariant spaces are obtained, and a description of the optimal rearrangement-invariant space for such an embedding is given; Conditions for the boundedness of a supremal operator in a weighted Lorentz space are obtained.

The obtained results develop the corresponding results of M.L. Goldman, A.Gogatishvili, A.Cianchi, R.Kerman, B.Opic, L.Pick, E.Bakhtygareeva, N.A. Bokaeva, G.Zh. Karshygina and others.

**Theoretical and practical value of the work.** The scientific results of the work are theoretical. Based on the obtained estimates for a nonincreasing rearrangement of a generalized fractional maximal function, various cones of monotonically decreasing functions are considered. As a result, criteria for embedding the space of generalized fractional maximal functions into rearrangement-invariant spaces are obtained. A comparison is made between the generalized fractional maximal function and the generalized Riesz potential.

The obtained results can be used in the study of other operators in rearrangement-invariant spaces.

The study of the integral properties of the generalized fractional maximal function serves as the basis for further study of the smoothness properties of the function in other integral metrics.

The obtained scientific results can be used in special courses for students, master's and doctoral students.

**Provisions submitted for presentation.** The following main results of the dissertation research are submitted for defense:

1) a generalized fractional maximal function is determined, which in a certain case corresponds to the classical fractional maximal function;

2) different estimates were obtained for a non-increasing rearrangement of the generalized fractional maximal function;

3) various cones associated with a non-increasing rearrangement of functions were constructed, and conditions for mutual covering of such cones were obtained;

4) conditions for embedding the space of generalized fractional maximal functions into rearrangement-invariant spaces are obtained, and a description of the optimal rearrangement-invariant space for such an embedding is given;

5) conditions for the boundedness of a supremal operator in a weighted Lorentz space are obtained.

**The credibility and validity** of the research carried out is ensured by the proposed proofs of the presented theorems and lemmas, and substantiated by the cited publications.

**Internal unity of the dissertation work.** The internal unity of the dissertation work is achieved by the purpose and objectives of the research. The main results obtained in the dissertation made it possible to determine the scientific novelty of the theory of operators in function spaces. The condition for the internal unity of this work is the study of the relationship between various cones generated by a non-increasing rearrangement of generalized fractional maximal functions and the embedding of such function spaces in rearrangement-invariant spaces.

**Approbation of the obtained results.** The results of the dissertation were presented and discussed at the following seminars and conferences:

1. The XVI International Scientific Conference for students and young scholars «Ǵylym jáne bilim – 2021», L.N. Gumilyov ENU, April 12, 2021. (Nur-Sultan)
2. «Algebra, Topology and Analysis: $C^*$ and $A_\infty$ algebras», Summer School Gonio (Batumi, 30.08.2021-03.09.2021).
3. Annual international April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan (Almaty: Institute of Mathematics and Mathematical Modeling, 2022).
4. The XVII International Scientific Conference for students and young scholars «Ǵylym jáne bilim – 2022», L.N. Gumilyov ENU, April 12, 2022 (Nur-Sultan).
5. IX International Scientific Conference «Problems of Differential Equations, Analysis and Algebra», Aktobe: K. Zhubanov Aktobe Regional University, May 24-28, 2022.
6. "Actual problems of mathematics, mechanics and informatics" International scientific conference dedicated to the 80th anniversary of Professor T.G. Mustafina (Karaganda: Karaganda Buketov University, September 8-9, 2022).
7. International Conference on Mathematics and Mathematics Education (ICMME - 2022) (Denizli: Pamukkale University, 2022).
8. Annual international April Mathematical Conference in honor of the Day of Science Workers of the Republic of Kazakhstan (Almaty: Institute of Mathematics and Mathematical Modeling, 2023).
9. XVIII International Scientific Conference of Students, Masters and Young scholars "Lomonosov–2023", April 14-15, 2023 (Astana).
10. The XVIII International Scientific Conference for students and young scholars «Ǵylym jáne bilim – 2023», L.N. Gumilyov ENU, April 12, 2023 (Astana).
11. 6th international Hybrid conference on Mathematical Advances and Applications (ICOMAA-2023) (Istanbul: Yildiz Technical University, 2023).
12. International scientific and practical conference "Analysis, differential equations and their applications" dedicated to the 100th anniversary of the birth of T.I. Amanov (Astana: Kazakhstan branch of M.V. Lomonosov Moscow State University, ENU named after. L.N. Gumilyov, 2023).
13. VII World Congress of Mathematicians of the Turkic World (TWMS Congress-2023), September 20-23, 2023 (Turkestan).

Also the results of the dissertation were presented and discussed at scientific seminars of the Department of Fundamental Mathematics of ENU named after. L.N. Gumilyov:

14. Scientific seminar "Functional analysis and its applications", Leaders: Academician of the NAS RK R.O. Oinarov, E.D. Nursultanov, K.N. Ospanov, October 19, 2023, November 2, 2023, November 16, 2023.

**Publications.** 19 works (4 articles, 15 theses) were published on the topic of the dissertation: 2 of them were published in journals indexed in the Scopus database ("Eurasian Mathematical Journal", percentile – 49, "Bulletin of the Karaganda

University. Mathematics Series", percentile – 35) , 2 articles were published in journals recommended by the Committee for Quality Assurance in the Field of Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan and 15 works were published in materials of international scientific conferences.

In works carried out in co-authorship, the author was directly involved at all stages of work related to the preparation of articles, under the guidance of scientific consultants. The author personally contributed to the achievement of the main results.

**Structure and scope of work.** The work consists of an introduction, two sections consisting of subsections, a conclusion and a list of sources used.

The numbering of formulas consists of three indicators. The first index indicates the section number, the second index indicates the order of the section's subsections, and the third index indicates the order of the formulas in this subsection. The total volume of the dissertation is 82 pages.

**Number of sources used** – 53.

**Keywords.** Generalized fractional maximal function, generalized Riesz potential, function space, nonincreasing rearrangements, cones, rearrangement-invariant spaces, embedding theorems.