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SOLVABILITY OF MIXED PROBLEMS FOR THE WAVE AND HEAT EQUATIONS WITH INVOLUTION

ABSTRACT

of the thesis for the degree of Doctor of Philosophy (PhD) in the educational program 8D05410 – Mathematics

Urgency of the research. Research on the spectral theory of ordinary differential operators started with the classical works of J. Liouville, S. Sturm, V.A. Steklov, Yu.D. Tamarkin and other authors. Further development of the spectral theory of differential operators defined two branches of the spectral theory: the spectral theory of self-adjoint operators and the spectral theory of non-self-adjoint operators.

By the present time, the spectral theory of self-adjoint differential operators has ben studied quite well, and the most important problem of spectral theory, the question of the basis property of a system of eigenfunctions, has been solved. It is known that the system of eigenfunctions of a self-adjoint differential operator with discrete spectrum forms a completely orthonormal system, which is an orthonormal basis.

The study of the spectral properties of non-self-adjoint ordinary differential operators required new approaches. G. Birkhoff introduced the concept of regular boundary conditions and proposed the Cauchy integral approach. The approach is based on estimating the Green's function of boundary value problems, and it is required that the eigenvalues be single. Many works were devoted to the study of problems with regular boundary conditions. Later it was revealed that among the regular boundary conditions for ordinary differential operators of even order there are the so-called strongly regular boundary conditions with special properties. In the works of G. Keselman and V. P. Mikhailov, it was established that the system of eigenfunctions of ordinary differential operators of odd order with regular boundary conditions and the system of eigenfunctions of ordinary differential operators of even order with strongly regular boundary conditions form the Riesz basis.

A specific feature of strongly regular boundary conditions for ordinary differential operators of even order is that all eigenvalues are simple. If the boundary conditions are regular but not strongly regular, the operator can have infinitely many multiple eigenvalues. The study of the basic properties of the system of eigenfunctions of ordinary differential operators with multiple eigenvalues turned out to be a difficult task. For example, the problem of the basis property of a system of eigenfunctions of the second-order ordinary differential operators with periodic boundary conditions has not yet been solved. However, V.A. Il'in proposed a new approach to the study of the basic properties of the system of eigenfunctions of ordinary differential operators with multiple eigenvalues.

This approach does not depend on the type of boundary conditions, but is based on the formula that determines the average value of the solution to the equation, and the basis property conditions are formulated as a product of the norms of root functions, direct and adjoint spectral problems. In the course of the investigations on the basis property of eigenfunctions, described in Chapter 1, it was found that these methods complement each other. It should be noted that differential equations with involution were first studied in the works of A. Babbage in the 19th century. Studies on various issues in the theory of differential equations with involution can be found in the monographs of D. Przeworska-Rolewicz, J. Wiener, A. Cabada and F.A.F. Tojo. Despite the fact that the study of involutional differential equations has a fairly long history, spectral problems were not considered until the works of T.Sh. Kalmenov and A. Shaldanbaev. Spectral problems for differential equations with involution in the second derivative were developed in the studies of A.M. Sarsenbi, A. Tengaeva, M.A. Sadybekov and A.M. Sarsenbi, A. Kopzhasarova and A.M. Sarsenbi, L.V. Kritskov and A.M. Sarsenbi, L.V. Kritskova, M.A. Sadybekov and A.M. Sarsenbi, L.V. Kritskova and V.L. Ioffe, A.A. Sarsenbi, A.A. Sarsenbi and B.Kh. Turmetova, A.A. Sarsenbi and A.M. Sarsenbi. Spectral problems for differential equations of higher orders with involution in lower terms are also presented in the works of Yu.A. Baranetsky and L. Kolyasa. In the works by A.G. Baskakov and his disciples, A.P. Khromov and his disciples, spectral problems for involutional differential equations of the first order are studied.

The results of research by A.Kopzhasarova and A.M. Sarsenbi were applied to the study of direct and inverse problems for the wave equation, the heat equation with involution and their fractional analogues.

Involution differential equations occupy a special place among differential and functional-differential equations with argument reflection. Differential equations with involution have a wide range of applications in various fields of science. When studying geometric problems, the use of equations with an involution of the form $f(x)=c-x$ was implemented in the work of S.F. lacroix. In the works of R. Bellman and K. L. Cooke, the problem of I. Bernoulli and L. Euler is studied by reducing it to a differential equation with an involution of the form $f(x)=-x$. Such involutional transformations are used in classical statistical mechanics, filtration theory and other fields. Information on mathematical models describing the phenomena of physiology, ecology, and population can be found in the above-mentioned monographs by D.J. Wiener.

A special case of the equations with involution studied in the thesis are the well-known classical Sturm-Liouville equations, heat conduction and wave equations. The interest of world scientists in the study of involutional differential equations confirms the importance and relevance of the topic of the thesis.

Work objective. Investigation of the basic properties of systems of eigenvectors of the second-order differential operators with involution, whose coefficients are complex-valued functions, as well as study of solvability of mixed

problems for the heat equation and the wave equation with involution, whose coefficients are complex-valued functions.

Object of study. Second-order differential equations with involution whose coefficients are complex-valued functions; heat conduction equations and a wave equation with involution whose coefficients are complex-valued functions.

Research methods. The author uses the Cauchy's integral method, Fourier's method, analytical methods of the theory of differential equations, methods of the abstract theory of linear operators in Hilbert space, spectral theory of differential operators, and methods of functional analysis.

Научная новизна исследования.

Scientific novelty of the research.

The following main scientific results were obtained in the thesis.

1 For second-order differential equations with involution whose coefficients are complex-valued functions:

a) the location of the eigenvalues of the Dirichlet, Neumann problems, periodic and antiperiodic boundary value problems on the complex plane is determined;

b) the equiconvergence theorem for the antiperiodic boundary value problem is proved, it is also shown that the system of eigenfunctions forms a basis in the space $L_2(-1,1)\, L_2(-1,1)$;

c) it is proved that the system of eigenfunctions of the Dirichlet, Neumann, periodic and antiperiodic boundary value problems forms the Riesz basis in the space $L_2(-1,1)$.

2 Existence and uniqueness theorems for solutions of mixed problems are proved for a wave equation with involution whose coefficients are complex-valued functions.

3 Existence and uniqueness theorems for the solution of the inverse problem for the heat equation with involution, whose coefficients are complex-valued functions, are proved.

The spectral properties of the second-order differential equations with involution are studied. The Green's function of the antiperiodic boundary value problem is constructed and a uniform estimate of the Green's function is obtained. Based on estimates of the Green's function, the results on the completeness and basis property of the desired system of eigenfunctions were obtained. The conditions for the basis property and unconditional basis property for eigenfunctions in the space $L_2(-1,1)$ are obtained. The existence and uniqueness theorems for the solution of mixed and inverse problems for the wave equation and for the heat equation with a variable coefficient, respectively, are proved.

Approbation of the obtained results. The results of the thesis were presented at the following conferences:

1. Traditional international April mathematical conference in honor of the Day of Science Workers (Almaty, 2021, 2023).

2. 6th International Conference on Mathematical Sciences, 2022, July 20-24, Istanbul, Turkey.

In addition, the results of the work were discussed at the following seminar:

1. Scientific Center "Theoretical and Applied Mathematics". City scientific seminar, Shymkent, 2023

Publications

12 papers based on the research materials were published, including 1 article in the scientific journal recommended by the Committee, 2 articles in the journals included in the Q1 quartile according to the Thomson Reuters information base (ISI Web of Knowledge, Thomson Reuters) and 9 articles in the proceedings of international conferences, including 3 articles published in international conference materials.

In particular, the following articles were published in the rating journals indexed in the Web of Science and Scopus databases:

1. Mussirepova E., Sarsenbi A.M., Sarsenbi A.A. Solvability of mixed problems for the wave equation with reflection of the argument // Math Meth Appl Sci. 2022; 45:11262–11271, DOI:10.1002/mma.8448

2. Mussirepova E., Sarsenbi A.M., Sarsenbi A.A. The inverse problem for the heat equation with reflection of the argument and with a complex coefficient // Boundary Value Problems (2022) 2022:99 https://doi.org/10.1186/s13661-022-01675-1

Articles published in the journals included in the list of the CQAFSHE (Committee for Quality Assurance in the Field of Science and Higher Education) MSHE RK:

1. Sarsenbi A.A., Mussirepova E. Green's function of a boundary value problem for a second-order differential equation with involution // Қазақстан Республикасы Ұлттық инженерлік академиясының хабаршысы, Алматы 2022г., No.4 (86)

Articles published in international conference materials:

1. Сәрсенбі Ә.М., Мүсірепова Э. Сәрсенбі Ә.М., Мүсірепова Э. Eigenvalues of a boundary value problem with involution // VI Oraz Readings: Topical Issues of Turkic Culture in the Digital Age. International scientific and practical conference. Shymkent, SWIU. 2020, V. III, pp. 202-209.

2. Мүсірепова Э., Жорғабаев Ғ. Оң анықталған инволюциясы бар екінші ретті дифференциалды операторлар // IV Yunus readings: Trends and prospects for the development of education and science in the context of international integration. Shymkent, SWIU. 2021, pp. 399-401.

3. Мүсірепова Э., Султанбек Д. Инволюциясы бар екінші ретті дифференциалды теңдеу үшін Дирихле шеттік есебінің меншікті мәндері // IV Yunus readings: Trends and prospects for the development of education and science in the context of international integration. Shymkent, SWIU. 2021, pp. 399-401.

4. Sarsenbi A.M., Mussirepova E., Sarsenbi A.A. Existence and uniqueness of solution to wave equation with involution // 6-th international conference of mathematical sciences, ICMS 2022, Istanbul, Turkey

5. Mussirepova E., Sarsenbi A.A. Green’s function of a boundary value problems for a differential equation with involution // 6-th international conference of mathematical sciences, ICMS 20227., Istanbul, Turkey

6. Mussirepova E., Sarsenbi A.M., Sarsenbi A.A. Existence and uniqueness solution of wave equations with involution // Sixth International Conference on Analysis and Applied Mathematics, Bahcesehir University, Turkey
7. Мүсірепова Э., Сарсенби А.А., Сарсенби А.М. Solvability of mixed problems for the wave equation with involution // ACTUAL PROBLEMS OF MATHEMATICS, MECHANICS AND COMPUTER SCIENCE. Proceedings of the International scientific conference dedicated to the 80th anniversary of Professor T.G. Mustafin.
8. Мүсірепова Э., Сарсенби А.А., Сарсенби А.М. Solvability of mixed problems for the heat equation with involution // ACTUAL PROBLEMS OF MATHEMATICS, MECHANICS AND COMPUTER SCIENCE Proceedings of the International scientific conference dedicated to the 80th anniversary of Professor T.G. Mustafin.
9. Сарсенби А.А., Мүсірепова Э. Eigenfunctions of an antiperiodic problem for a differential equation with involution // Traditional international April mathematical conference in honor of the day of science of the Republic of Kazakhstan. Abstracts of reports. Almaty, IM&MM. 2023, p. 106

The structure and volume of the thesis. The thesis consists of an introduction, two parts (each part is divided into paragraphs), a conclusion and a list of references, consisting of 70 titles. The total volume of the dissertation is 83 pages.

In the final part, the main results obtained in the thesis are briefly described and the possibilities of their application are shown.