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Parabolic Problems in Noncylindrical Domains with Special Boundary Conditions

Thesis abstract for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Thesis structure. The Thesis work consists of an introduction, two sections, a conclusion, a list of references and an appendix. The numbering of formulas, theorems, lemmas, statements and remarks in sections is three-digit, the first number means the number of the section, the second - the number of the subsection, the third - its own number within the subsection.

Keywords. Boundary value problems for the heat equation, heat potential, singular Volterra integral equations of the second kind, Laplace transform, resolvent, domain degenerating into a point.

Relevance of the topic. The need to solve boundary value problems for the equations of nonstationary transport in domains with boundaries that change with time is explained by the fact that they have a wide practical application. Problems of this kind describe electromagnetic, gas-dynamic, and thermophysical processes in low- and high-pressure gas-discharge plasmas. Mathematical modeling is indispensable for the development of plasma installations, as it provides the necessary information about the optimal sizes and values of the fundamental parameters of processes and devices. They also arise when studying the processes of melting electrical contacts, the effect of an electric arc on contacts; when studying the problems of thermal shock in domains with a moving boundary, when solving a number of problems in hydromechanics. Problems of this kind are of great practical value for studying thermal effects during crack propagation, which leads to the destruction of materials and mechanisms; when studying the freezing of solutions, soils; in the study of the kinetic growth of crystals.

An experimental study of such phenomena, in domains degenerating into a point at the initial moment of time, is difficult due to their transience and, in most cases, only a mathematical model can serve as a basis for obtaining additional information about their dynamics. Analytical methods in this case provide an opportunity for a visual and convenient analysis of phenomena, allow you to reflect the influence of all factors, evaluate their significance and highlight the main ones. In addition, the presence of analytical solutions of a certain class of boundary value problems is also of interest for constructing difference schemes for the approximate calculation of solutions to rather complex problems. However, the complexity of the analytical solution of heat conduction problems in domains with boundaries that change with time and degenerate to a point is determined by the fact that classical methods of differential equations of mathematical physics are not directly applicable to this type of problems. The peculiarity of the studying such problems is that when the size of the domain depends on time and the domain degenerates into a point at the initial moment of time, it is not possible to coordinate the solution of the equation with the motion of the domain boundaries. Finding analytical solutions for the indicated classes of heat conduction problems requires special methods or modifications of known approaches.

Therefore, the issue of studying boundary value problems in domains with degeneracy at the initial moment of time with special boundary conditions is not fully studied theoretically and, accordingly, is relevant.

Purpose of research – Statement and solution of boundary value problems for heat conduction equations with special boundary conditions in non-cylindrical domains degenerating into a point at the initial moment of time; solution of singular Volterra type integral equations of the second kind; study of their solvability.

Research objectives:

- to give the statement of new boundary value problems for heat conduction equations in non-cylindrical, degenerate domains with special boundary conditions and describe the spaces of solutions and given functions;
- transformation of initial problems;
- reduction of boundary value problems to the singular Volterra integral equations of the second kind ;
- solution of special integral equations, construction resolvents;
- solution of initial boundary value problems;
- definition of uniqueness classes for the studied boundary value problems;

Research object: boundary value problems for equations of parabolic type with special boundary conditions in noncylindrical domains that degenerate into a point at the initial moment of time.

Research subject: solvability of boundary value problems for heat conduction equations with special boundary conditions in domains that degenerate to a point at the initial moment of time and the solution of related singular Volterra type integral equations of the second kind.

Research methodology. The Thesis uses the methods of the general theory of differential equations and functional analysis, the methods of Laplace integral transformations, the theory of special functions and the theory of functions of a complex variable.

Scientific novelty. The paper proposes the statement of new boundary value problems in non-cylindrical domains for the heat equation with special boundary conditions. The peculiarities of the problems under consideration lead to the fact that it becomes necessary to study the solvability of the singular Volterra type integral equations of the second kind.

Theoretical and practical value of the work. The results of the thesis are theoretical. It developed a method for studying a number of boundary value problems for heat conduction equations in non-cylindrical domains, with special boundary conditions, based on the reduction of the studied problems to the singular Volterra type integral equations of the second kind. The solution of singular integral equations is obtained in a closed form.

In addition, the results obtained can serve as a certain contribution to the theory of integral equations with variable limits of integration with singularities of the kernel. The practical value of the work is determined by the fact that it is useful in the study of some problems with free boundaries, for example, in the study of the single-phase Stefan problem.

Highlights for defense. The following highlights are defended:

1^0 Special boundary value problems for the heat equation in weighted functional classes and their equivalent transformations;

2^0 Equivalence of boundary value problems constructed by the singular Volterra type integral equations of the second kind;

3^0 Construction of the resolvent of the singular Volterra type integral equations of the second kind;

4^0 Spectral properties of the singular Volterra type integral operators of the second kind: an explicit representation of the eigenfunctions is found;

5^0 Weighted classes of uniqueness for the studied boundary value problems.

Reliability and validity of the conducted research are provided with constructiveness of the developed and used methods. Auxiliary statements of the problematic issues touched upon in each section are formulated in the form of lemmas and statements, and they are rigorously proved, while general statements are in the form of theorems and their proofs are presented in detail.

Publications. The main results of the dissertation are published in: 4 papers and 7 theses. Of these, 1 paper is in the journal included in the Scopus database (48% percentile).

In papers performed with co-authors, the contribution of each of the co-authors is equal.

In **subsection 1.1** of the first section, the following boundary value problem of heat conduction is considered. In the domain $G = \{x, t|\, 0 < x < t^\omega,\ t > 0\}$, we study the solvability issues of the boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{1}$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = u_0(t),\ \frac{d\tilde{u}\,(t)}{dt} + \left.\frac{\partial u}{\partial x}\right|_{x=t^\omega} = u_1(t). \tag{2}$$

where $\tilde{u}\,(t) = u\,(t^\omega, t)\,, \omega > 1/2$.

We introduce the classes of solutions and data of the problem as follows:

$$\left(x + t^{\frac{3}{2}-\omega}\right)^{-1} u(x,t) \in L_\infty(G) \text{ т.е. } u(x,t) \in L_\infty\left(G; \left(x + t^{\frac{3}{2}-\omega}\right)\right)$$

$$f(x,t) \in W_\infty^{1,0}\left(G; t^{\frac{3}{2}-\omega} \exp\left(\frac{t^{\frac{3}{2}-\omega}}{4a^2}\right)\right); \tag{3}$$

$$u_0(t) \in L_\infty\left(R_+; \left(t^{-\left(\frac{3}{2}-\omega\right)}\right)\right); \ u_1(t) \in L_\infty\left(R_+; \left(t^{\frac{3}{2}-\omega}\right)\right)$$

This kind of boundary value problem (1)-(2) arises, for example, in studies of the Stefan problem.

Introducing a new unknown function $v(x,t) = \frac{\partial u}{\partial x}$, we transform problem (1) –(2) to the next problem:

$$\frac{\partial v}{\partial t} - a^2 \frac{\partial^2 v}{\partial x^2} = \widetilde{f}(x,t),\ \ 0 < x < t,\ t > 0 \tag{4}$$

$$v|_{x=0} = v_0(t),\ \ \left.\left(\frac{\partial v}{\partial x} + \frac{1 + \omega t^{\omega-1}}{a^2} v\right)\right|_{x=t^\omega} = v_1(t) \tag{5}$$

where $\tilde{f}(x,t) \equiv \frac{\partial f(x,t)}{\partial x}$, $v_0(t) \equiv u_0(t)$, $v_1(t) \equiv \frac{u_1(t)}{a^2} + \left.\frac{f}{a^2}\right|_{x=t^\omega}$

Remark 1 *Each solution to boundary value problem* (4) - (5) *defines a unique solution (up to a constant) of boundary value problem* (1) –(2).

We will find the solution of problem (4)-(5) as the sum of heat potentials is sought in the form of the sum of the double- and simple-layer potentials , as well as the volume potential:

$$v(x,t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\xi)^2}{4a^2(t-\tau)}\right\} \tilde{f}(\xi,\tau)d\xi d\tau +$$

$$+ \frac{1}{4a^3\sqrt{\pi}} \int_0^t \frac{x}{(t-\tau)^{3/2}} \exp\left\{-\frac{x^2}{4a^2(t-\tau)}\right\} \nu(\tau)d\tau +$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \frac{1}{(t-\tau)^{1/2}} \exp\left\{-\frac{(x-\tau^\omega)^2}{4a^2(t-\tau)}\right\} \varphi(\tau)d\tau, \tag{6}$$

The function defined by equality (6) satisfies the equation (4) for any functions $\nu(t)$ and $\varphi(t)$, which are still unknown and are to be determined further.

Satisfying the boundary conditions, we obtain the following integral equation

$$\varphi(t) + \int_0^t K_\omega(t,\tau)\varphi(\tau)d\tau = F(t), \tag{7}$$

where kernel $K_\omega(t,\tau)$ can be represented as a sum:

$$K_\omega(t,\tau) = \sum_{i=1}^{4} K_\omega^{(i)}(t,\tau),$$

where:

$$\begin{aligned}
K_\omega^{(1)} &= \frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega + \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2\,(t-\tau)}\right\};\\
K_\omega^{(2)} &= -\frac{1}{2a\sqrt{\pi}} \cdot \frac{t^\omega - \tau^\omega}{(t-\tau)^{3/2}} \cdot \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2\,(t-\tau)}\right\};\\
K_\omega^{(3)} &= -\frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega+\tau^\omega)^2}{4a^2\,(t-\tau)}\right\};\\
K_\omega^{(4)} &= \frac{1}{a\sqrt{\pi}} \cdot \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \exp\left\{-\frac{(t^\omega-\tau^\omega)^2}{4a^2\,(t-\tau)}\right\}.
\end{aligned} \tag{8}$$

The free term of equation (7) has the following form

$$F(t) = -\frac{a}{\sqrt{\pi}} \int\limits_0^t \left[\frac{1}{(t-\tau)^{3/2}} - \frac{t^{2\omega}}{2a^2(t-\tau)^{\frac{5}{2}}}\right] \exp\left\{-\frac{t^{2\omega}}{4a^2(t-\tau)}\right\} v_0(\tau)d\tau +$$

$$-\frac{1+\omega t^{\omega-1}}{a\sqrt{\pi}} \int\limits_0^t \frac{t^\omega}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{t^{2\omega}}{4a^2(t-\tau)}\right\} v_0(\tau)d\tau + 2a^2 \cdot v_1(t) - \frac{1}{2a\sqrt{\pi}} \times$$

$$\times \int\limits_0^t \int\limits_0^\infty \left[\frac{t^\omega+\xi}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(t^\omega+\xi)^2}{4a^2(t-\tau)}\right\} - \frac{t^\omega-\xi}{(t-\tau)^{\frac{3}{2}}} \exp\left\{-\frac{(t^\omega-\xi)^2}{4a^2(t-\tau)}\right\}\right] \tilde{f}(\xi,\tau)d\xi d\tau -$$

$$-\frac{1}{a\sqrt{\pi}} \int\limits_0^t \int\limits_0^\infty \frac{1+\omega t^{\omega-1}}{(t-\tau)^{1/2}} \cdot \exp\left\{-\frac{(t^\omega-\xi)^2}{4a^2(t-\tau)}\right\} \cdot \tilde{f}(\xi,\tau)d\xi d\tau. \tag{9}$$

We will find the solution of integral equation (7) in the class of functions:

$$t^{\frac{3}{2}-\omega} \cdot \varphi(t) \in L_\infty(0,\infty), \ \text{ i.e. } \ \varphi(t) \in L_\infty\left(0,\infty;\ t^{\frac{3}{2}-\omega}\right). \tag{10}$$

For convenience, we represent equation (7) as follows:

$$\varphi_1(t) + \int_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} K_\omega(t,\tau)\varphi_1(\tau)d\tau = F_1(t). \tag{11}$$

where

$$\varphi_1(t) = t^{\frac{3}{2}-\omega} \cdot \varphi(t), \quad F_1(t) = t^{\frac{3}{2}-\omega} \cdot F(t) \tag{12}$$

We note that the kernel $K_\omega(t,\tau)$ has the properties:

1) $K_\omega(t,\tau)$ is continuous when $0 < \tau \le t \le \infty$;

2) $\lim\limits_{t\to t_0} \int\limits_{t_0}^{t} K_\omega(t,\tau)d\tau = 0, \ \ t_0 \ge \varepsilon > 0$;

3) $\lim\limits_{t\to 0+} \int\limits_{0}^{t} K_\omega(t,\tau)d\tau = 1$.

The singularity of the integral equation (11) is property 3 of kernel $K_\omega(t,\tau)$.

In order to solve integral equation (11) we will construct the corresponding characteristic integral equation.

$$\varphi(t)+\int\limits_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}K_h(t,\tau)\cdot\varphi(\tau)\,d\tau=g(t), \tag{13}$$

where

$$K_h(t,\tau)=\sum_{i=1}^4 K_h^{(i)}(t,\tau),$$

$$K_h^{(1)}(t,\tau)=\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{\frac{3}{2}}\left(\tau^{2\omega-1}\cdot t^{2\omega-2}+t^{4\omega-3}\right)}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{\frac{3}{2}}}\cdot\exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$
$$K_h^{(2)}(t,\tau)=-\frac{1}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{3/2}\cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}}\cdot\exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$
$$K_h^{(3)}(t,\tau)=-\frac{2}{2a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{3/2}\cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}}\cdot\exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}+\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\};$$
$$K_h^{(4)}(t,\tau)=\frac{2}{a\sqrt{\pi}}\cdot\frac{(2\omega-1)^{3/2}\cdot t^{2\omega-2}}{\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^{1/2}}\cdot\exp\left\{-\frac{(2\omega-1)\left(t^{2\omega-1}-\tau^{2\omega-1}\right)^2}{4a^2\left(t^{2\omega-1}-\tau^{2\omega-1}\right)}\right\}. \tag{14}$$

Integral equation (13) it is indeed characteristic equation for the equation (11), since its kernel has a property similar to the property 3 of kernel $K_\omega(t,\tau)$.

$$\lim_{t\to 0}\int\limits_0^t K_h^{(1)}(t,\tau)\,d\tau=1.$$

Hence, it follows that

$$\lim_{t\to 0}\int\limits_0^t\left[K_h(t,\tau)-K_\omega(t,\tau)\right]d\tau=0,\quad 0<\tau<t<\infty. \tag{15}$$

The solution of the characteristic equation (13) is found :

$$\varphi(t)=g(t)+\int\limits_0^t\left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega}\cdot R_h(t,\tau)\cdot g(\tau)\,d\tau+C\cdot\varphi_0\left((2\omega-1)\cdot t^{2\omega-1}\right). \tag{16}$$

For the resolvent $R_h(t,\tau)$ the following Lemma holds.

Lemma 0.1

$$R_h(t,\tau)\le C_1(\omega)\cdot\frac{t^{\omega-\frac{1}{2}}\cdot\sqrt{\tau}}{(t-\tau)^{\frac{3}{2}}}\cdot exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{(2\omega-1)\cdot a^2\cdot(t-\tau)}\right\}. \tag{17}$$

The following Theorem proved.

Theorem 0.1 *For any right side* $g(t) \in L_\infty\left(\mathrm{R}_+; t^{\frac{3}{2}-\omega}\right)$ *integral equation* (13) *has a general solution* $\varphi(t) \in L_\infty\left(\mathrm{R}_+; t^{\frac{3}{2}-\omega}\right)$*:*

$$\varphi(t) = g(t) + \int\limits_0^t \left(\frac{t}{\tau}\right)^{\frac{3}{2}-\omega} \cdot R_h(t,\tau) \cdot g(t)\, d\tau + C \cdot \varphi_{hom}\left((2\omega-1)\cdot t^{2\omega-1}\right), \qquad (18)$$

and for the resolvent $R_h(t,\tau)$ *we have estimate*

$$R_h(t,\tau) \le C_1(\omega) \cdot \frac{t^{\omega-\frac{1}{2}} \cdot \sqrt{\tau}}{(t-\tau)^{\frac{3}{2}}} \cdot exp\left\{-\frac{t^{2\omega-1}\cdot\tau}{(2\omega-1)\cdot a^2\cdot(t-\tau)}\right\}.$$

To solve the integral equation (7), the Carleman-Vekua regularization method is used.

For the original boundary value problem (1) –(2) is formulated in the following theorem:

Theorem 0.2 *For any right side* $f(t) \in L_\infty\left(\mathrm{R}_+;\ t^{\frac{3}{2}-\omega}\exp\left\{\frac{t^\omega}{4a^2}\right\}\right)$ *and for given functions* $f(x,t) \in W_\infty^{1,0}\left(G; t^{\frac{3}{2}-\omega}\exp\left\{\frac{t^{2\omega-1}}{4a^2}\right\}\right)$, $u_0(t) \in L_\infty(R_+; t^{\omega-\frac{3}{2}})$; $u_1(t) \in L_\infty(R_+;\ t^{\frac{3}{2}-\omega})$ *boundary value problem* (1) – (2) *has a general solution* $u(x,t) \in L_\infty(G; \left(x + t^{\frac{3}{2}-\omega})^{-1}\right)$.

In subsection 1.2. the same boundary value problem for the heat equation is considered as in Subsection 1.1. However, the difference is that the boundary of the domain moves according to an arbitrary law $x = \gamma(t)$,, while in the previous problem the boundary moved according to a power law. The study of both problems is carried out according to the same scheme, however, the case of a power-law motion of the boundary is considered separately, since the essence of the solution method in this case can be traced more clearly. Therefore, it should be considered that the results of these two cases methodically complement each other. This served as the basis for a separate consideration and a separate presentation of the results obtained in this cases.

In **subsection 1.1** of the first section, the following boundary value problem of heat conduction is considered.

In the domain $G = \{x,t|\, 0 < x < \gamma(t),\ t > 0\}$, we study the solvability issues of the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad \{0 < x < \gamma(t),\ t > 0\} \qquad (19)$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = u_0(t), \quad \frac{d\tilde{u}(t)}{dt} + \left.\frac{\partial u}{\partial x}\right|_{x=\gamma(t)} = u_1(t), \qquad (20)$$

where $\tilde{u}(t) = u(\gamma(t),t)$, $\gamma(0) = 0$ for $\gamma(t) = [t(1+\alpha_0(t))]^\omega$, $\omega > \frac{1}{2}$

Function $\gamma(t) : (0,\infty) \to (0,\infty)$ satisfies the following conditions:

1. asymptotics of the function $\gamma(t)$ as $t \to 0$ and as $t \to \infty$ has the form t^ω, where $\omega > \frac{1}{2}$

2. starting from some moment of time t_1^* until moment of time t_2^* the function $\gamma(t)$ is arbitrary, strictly monotone and one-to-one, i.e. there is a reverse transformation $\gamma^{-1}(t)$.

We introduce the classes of solutions and data of the problem as follows:

$$(x + [\gamma(t)]^{\frac{3}{2\omega}-1})^{-1}u(x,t) \in L_\infty(G), \text{ т.е. } \ u(x,t) \in L_\infty(G; (x + [\gamma(t)]^{3/2\omega-1})^{-1}),$$

$$f(x,t) \in W_\infty^{1,0}\left(G; [\gamma(t)]^{3/2\omega-1}\exp\left\{[\gamma(t)]^{\frac{2\omega-1}{\omega}}/(4a^2)\right\}\right);$$

$$u_0(t) \in L_\infty\left(R_+; [\gamma(t)]^{-(3/2\omega-1)}\right); \ u_1(t) \in L_\infty\left(R_+; [\gamma(t)]^{3/2\omega-1}\right). \qquad (21)$$

For the boundary value problem (19) –(20) we proved the Theorem:

Theorem 0.3 *For any right side* $f(t) \in L_\infty\left(\mathrm{R}_+;\ [\gamma(t)]^{\frac{3/2-\omega}{\omega}} \exp\{\gamma(t)/(4a^2)\}\right)$ *and for given functions* $f(x,t) \in W_\infty^{1,0}\left(G; [\gamma(t)]^{3/2\omega-1} \exp\left\{[\gamma(t)]^{\frac{2\omega-1}{\omega}}/(4a^2)\right\}\right)$, $u_0(t) \in L_\infty(R_+; [\gamma(t)]^{\frac{\omega-3/2}{\omega}})$; $u_1(t) \in L_\infty(\mathrm{R}_+;\ [\gamma(t)]^{\frac{3/2-\omega}{\omega}})$ *boundary value problem* (19) – (20) *has a general solution* $u(x,t) \in L_\infty(G; \left(x + [\gamma(t)]^{3/2\omega-1}\right)^{-1})$.

In the *second* chapter of the work, we consider a two-dimensional boundary value problem in spatial variables in an inverted cone $Q = \left\{(x,y,t)|\ \sqrt{x^2+y^2} < t,\ 0 < t < 1\right\}$:

$$\frac{\partial u}{\partial t} - a^2 \triangle u = f(x,y,t), \tag{22}$$

$$\frac{d\tilde{u}}{dt} + \frac{\partial u}{\partial \bar{n}}\bigg|_{\sqrt{x^2+y^2}=t} = g(x,y,t), \tag{23}$$

where $\tilde{u}(t,\alpha) = u(x,y,t)|_{\sqrt{x^2+y^2}=t}$

Assuming that the axial symmetry condition is satisfied and passing in (22)–(23) to cylindrical coordinates, in the domain $G = \{(r,t)|\ 0 < r < t,\ 0 < t < 1\}$ we obtain the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = f(r,t), \tag{24}$$

$$\left(2\frac{\partial u}{\partial r} + \frac{\partial u}{\partial t}\right)\bigg|_{r=t} = g(t), \quad u(r,t) \neq \infty \qquad r \to 0. \tag{25}$$

In this case, the boundary condition (23) turns into the condition (25). Let's introduce a new function:

$$w(r,t) = r\frac{\partial u}{\partial r}, \qquad \left(\frac{\partial u}{\partial r} = \frac{1}{r}w(r,t)\right), \tag{26}$$

then the problem (24) - (25), is transformed into the following problem:

In the domain $Q = \{(r,t)|0 < r < t, 0 < t < 1\}$ find a solution of the equation

$$\frac{\partial w}{\partial t} = a^2\frac{\partial^2 w}{\partial r^2} - a^2\frac{1}{r}\frac{\partial w}{\partial r} + r\frac{\partial}{\partial r}f(r,t), \tag{27}$$

satisfying the boundary conditions:

$$\begin{cases} \left\{\dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{2}{a^2}\dfrac{1}{r}w\right\}\bigg|_{r=t} = g(t) - \dfrac{1}{a^2}f(t,t) = g_1(t), \\ \qquad\qquad\qquad\qquad w(r,t)|_{r=0} = g_2(t). \end{cases} \tag{28}$$

It is known that the function

$$G\left(r,\xi,t-\tau\right) = \frac{r}{2a\left(t-\tau\right)}\exp\left(-\frac{r^2+\xi^2}{4a^2\left(t-\tau\right)}\right)I_1\left(\frac{r\xi}{2a^2\left(t-\tau\right)}\right)$$

is the fundamental solution of the equation (27), ξ is a parameter. Hereinafter, $I_1(z)$ is the modified Bessel function of order 1.

We will seek the solution of the problem (27) - (28) as the sum of heat potentials: simple, double layer and volume potentials, that is, in the form

$$w(r,t) = \int_0^t G\left(r,\xi,t-\tau\right)|_{\xi=\tau}\,\mu(\tau)d\tau + \int_0^t \frac{\partial G(r,\xi,t-\tau)}{\partial \xi}\bigg|_{\xi=0}\nu(\tau)d\tau + W(r,t,f), \tag{29}$$

where

$$W(r,t,f)=\int_0^t d\tau\int_0^\tau G(r,\xi,t-\tau)r\frac{\partial f(\xi,\tau)}{\partial r}d\xi \tag{30}$$

is the heat volume potential. (It is a solution of the inhomogeneous equation (27)).

In the equality (29), In the equality $\mu(t)$ and $\nu(t)$ are potential densities which are still unknown functions.

Substituting the appropriate relations into the solution representation (29) we obtain the integral representation of the solution for the equation (27):

$$\begin{aligned}w(r,t)=\int_0^t\frac{r}{2a^2\,(t-\tau)}\exp\left(-\frac{r^2+\tau^2}{4a^2\,(t-\tau)}\right)I_1\left(\frac{r\tau}{2a^2\,(t-\tau)}\right)\mu(\tau)d\tau+\\+\int_0^t\frac{r^2}{8a^4(t-\tau)^2}\exp\left(-\frac{r^2}{4a^2(t-\tau)}\right)\nu(\tau)d\tau+W(r,t,f).\end{aligned} \tag{31}$$

Satisfying the boundary conditions, we obtain the following integral equation:

$$\begin{aligned}\mu_1(t)+\int_0^t\frac{t\tau}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau+\\+\int_0^t\frac{3}{2a^2}\frac{t}{t-\tau}\widetilde{I_1}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_1(\tau)d\tau=2a^2\mathcal{F}_1(t),\end{aligned} \tag{32}$$

where

$$\mathcal{F}_1(t)=-\left.\frac{\partial W(r,t)}{\partial r}\right|_{r=t}-\frac{2}{a^2}\left.W(r,t)\right|_{r=t}-\left.\frac{\partial\widetilde{g_2}(r,t)}{\partial r}\right|_{r=t}-\frac{2}{a^2}\left.\widetilde{g_2}(r,t)\right|_{r=t}-\frac{1}{a^2}tf(t,t)+g(t)$$

$$\widetilde{I_1}\left(\frac{r\tau}{2a^2(t-\tau)}\right)=\exp\left(-\frac{r\tau}{2a^2(t-\tau)}\right)I_1\left(\frac{r\tau}{2a^2(t-\tau)}\right).$$

$$\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)=\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right)\left[I_0\left(\frac{t\tau}{2a^2(t-\tau)}\right)-I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right],$$

$$\exp\left(\frac{\tau}{4a^2}\right)\mu(\tau)=\mu_1(\tau).$$

We introduce a new function

$$\mu_2(t)=t\mu_1(t)$$

then the equation (32) can be rewritten as follows:

$$\mu_2(t)+\int_0^t M(t,\tau)\mu_2(\tau)d\tau=2a^2t\mathcal{F}_1(t), \tag{33}$$

where

$$M(t,\tau)=M_1(t,\tau)+M_2(t,\tau), \tag{34}$$

here

$$M_1(t,\tau)=\frac{t^2}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right),$$

$$M_2(t,\tau)=\frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)}\widetilde{I_1}\left(\frac{t\tau}{2a^2(t-\tau)}\right),$$

Note the following property of the kernel (34), from which it follows that the method of successive approximations is not applicable to the integral equation(33).

Remark 2 $\lim\limits_{t\to 0}\int_0^t M(t,\tau)d\tau = 1,$ *and*

$$\int_0^t M_1(t,\tau)d\tau = 1, \ \ \forall t>0, \qquad lim_{t\to 0}\int_0^t M_2(t,\tau)d\tau = 0.$$

We will seek a solution of the following "truncated" integral equation, which, by Remark 2 is characteristic for the equation (33)

$$\mu_2(t) + \int_0^t \frac{t^2}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau = 2a^2 t\mathcal{F}_1(t). \tag{35}$$

In the integral equation (35) we change the independent variables: $t=\frac{1}{t_1}$, $\tau=\frac{1}{\tau_1}$ and by introducing the functions:

$$\mu_2\left(\frac{1}{t_1}\right) = \mu_2\left(t_1\right), \quad 2a^2\frac{1}{t_1}\mathcal{F}_1\left(\frac{1}{t_1}\right) = \mathcal{F}_2(t_1)$$

we obtain:

$$\mu_2(t_1) + \int_{t_1}^{\infty} \frac{1}{2a^2(\tau_1-t_1)^2}\widetilde{I_{01}}\left(\frac{1}{2a^2(\tau_1-t_1)}\right)\mu_2(\tau_1)d\tau_1 = \mathcal{F}_2(t_1), \tag{36}$$

This is an equation with a difference kernel, we apply the Laplace transform to both its sides, having previously written it in the following simplified form

$$\mu_2(t_1) + \int_{t_1}^{\infty} M_{1-}\left(t_1-\tau_1\right)\mu_2(\tau_1)d\tau_1 = \mathcal{F}_2(t_1), \tag{37}$$

where

$$M_{1-}\left(t_1-\tau_1\right) = \frac{1}{2a^2(\tau_1-t_1)^2}\widetilde{I_{01}}\left(\frac{1}{2a^2(\tau_1-t_1)}\right). \tag{38}$$

Then the solution of the equation (37) has the form

$$\mu_2(t_1) = \mathcal{F}_2(t_1) + \frac{1}{2\pi\imath}\int_{\sigma-\imath\infty}^{\sigma+\imath\infty}\widehat{R^*_-}(-p)\mathcal{F}_2(p)dp, \quad Rep<0 \tag{39}$$

where

$$\widehat{R^*_-}(-p) = \frac{1-2\frac{\sqrt{-p}}{a}I_0\left(\frac{\sqrt{-p}}{a}\right)K_1\left(\frac{\sqrt{-p}}{a}\right)}{2\frac{\sqrt{-p}}{a}I_0\left(\frac{\sqrt{-p}}{a}\right)K_1\left(\frac{\sqrt{-p}}{a}\right)} \tag{40}$$

Therefore, the solution of the equation (37) has the form

$$\mu_2(t_1) = \mathcal{F}_2(t_1) - \int_{t_1}^{\infty} R_-(t_1-\tau_1)\mu_2(\tau_1)d\tau_1,$$

where

$$\widehat{R^*_-}(-p) \risingdotseq R_-(t_1) = \frac{2a^2}{\pi^{\frac{3}{2}}\sqrt{t_1}}\sum_{k=1}^{\infty}B_k + \frac{2a^4}{\pi}\sum_{k=1}^{\infty}\alpha_k B_k \exp\left(-\alpha_k^2 a^4 t_1\right) -$$
$$-\frac{2a^2}{\pi^{\frac{3}{2}}\sqrt{t_1}}\sum_{k=1}^{\infty}\frac{N_1(\alpha_k)}{J_1(\alpha_k)}B_k\int_0^{\infty}\exp\left(-\frac{\xi^2}{4\alpha_k^2a^4t_1}\right)\sin\xi d\xi \tag{41}$$

where

Lemma 0.2 *For the resolvent $R_-(t_1)$ the following estimate holds*

$$R_-(t_1) \le C \cdot \frac{1}{\sqrt{t_1}}, \quad t_1 > 0.$$

The solution of the characteristic equation has the following form:

$$\mu_2(t) = 2a^2 t\mathcal{F}_1(t) - \int\limits_0^t \widetilde{R}(t,\tau)\mathcal{F}_1(\tau)d\tau$$

where

$$\widetilde{R}(t,\tau) \le C\frac{\sqrt{t}}{\sqrt{\tau}\sqrt{t-\tau}}.$$

To solve the "complete"integral equation (32)

$$\mu_2(t) + \int\limits_0^t \frac{t^2}{2a^2(t-\tau)^2}\widetilde{I_{01}}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau =$$

$$= -\int\limits_0^t \frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)}\exp\left(-\frac{t\tau}{2a^2(t-\tau)}\right) I_1\left(\frac{t\tau}{2a^2(t-\tau)}\right)\mu_2(\tau)d\tau + 2a^2 t \cdot \mathcal{F}_1(t), \quad (42)$$

we apply the regularization method by solving the characteristic equation - the Carleman-Vekua method.

Thus, the theorem holds.

Theorem 0.4 *If the conditions $\sqrt{t}g_1(t) \in L_\infty(0,1)$ and $g_2(t) \in L_\infty(0,1)$, are satisfied, then the boundary value problem (27)-(28) has a unique solution $w(r,t) \in L_\infty(G)$.*

From Theorem 0.4 and equality (31) we obtain the main result.

Theorem 0.5 *If $\sqrt{t}g(t) \in L_\infty(0,1)$, then boundary value problem (24)-(25) has a unique solution $u(r,t) \in L_\infty(G)$.*

The conclusion contains a brief summary on the results of Thesis research.

The Thesis ends with a list of references, appendix A, containing a list of published papers on the topic of the Thesis.